# Quantum Information in Geometric Quantum Mechanics

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**Abstract**

A fundamental starting point in quantum information theory is the consideration of the von Neumann entropy and its generalization to relative quantum entropies. A particular feature of quantum relative entropies is their relation via their Hessian to monotonic Riemannian metrics on the dense set of invertible mixed quantum states (Lesniewski and Ruskai 1999). These metrics are also known as quantum Fisher information metrics and provide a direct link to quantum estimation theory (Helstrom 1969). Quantum Fisher information metrics which are extendable to pure states coincide all with the Fubini Study metric of the projective Hilbert space of complex rays.

This theses outlines possible advantages of an inverse approach to quantum information theory, by starting with the Fubini Study metric rather then with the von Neumann entropy. This is done in a first step by associating to the Fubini Study metric a covariant and a contra-variant structure on the punctured Hilbert space as being available in the geometric formulation of quantum mechanics. While the contra-variant structure leads to a quantum version of the Cramér-Rao inequality for general 1-dimensional submanifolds of pure states, the covariant structure provides alternative entanglement monotones by identifying an inner product on the pullback tensor fields on local unitary group orbits of quantum states. It is shown in the case of two qubits that these monotones yield a more efficient estimation of entanglement than standard measures from the literature as those associated with the linearization of the von Neumann entropy.

*Per Daniela*

# Preface

This work focuses on mathematical foundations of quantum information from a geometric point of view. The main motivation for this project originated essentially from two recent observations made by Marmo and collaborators on the Fubini Study metric as used in covariant form in the geometric formulation of quantum mechanics. First, this metric links to alternative quantum entanglement measures [1–6], and second, it links to the so-called *quantum Fisher information* measure [7] as used in quantum estimation problems. The two concepts of entanglement measures and Fisher information are both well known as fundamental concepts in quantum information theory. However, *the connection between the two concepts in strict geometrical terms and it's resulting implications* hasn't been discussed yet and clearly indicates a lack on a deeper understanding on some of the most fundamental concepts of quantum information theory. The present work aims to fill this gap by reviewing the above stated two sets of observations in detail and by discussing what happens when merging these two together. As a result, a new application on the *quantum experimental bounds of weak entanglement quantification* within the currently emerging research field of *entanglement estimation theory* [8] is presented.

At this point I would like to thank all persons particularly involved in the process of this work.

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$a\,|\text{Napoli}\rangle + b\,|\text{München}\rangle$
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# 1 Quantum information – Getting started

In this first chapter we will give an introduction to the topic of the underlying work. For this purpose we will start with a brief overview on the basic ideas behind the concept of quantum information and its future emerging technological applications. The engineering problems arising here will then directly lead to the central motivation and content of this work.

## 1.1 Quantum information processing

In 1982, Feynman made following observation [9]. A computer simulation of the time evolution of a given quantum system requires a computational time *exponentially growing with the dimension* of the quantum system. Feynman concluded that, vice versa, given a computer based on the laws of quantum mechanics, one should in principle solve computational problems (including the problem of simulating the time evolution of a big quantum system) in a notable faster time.

The mathematical ingredient being responsible for the speed up may be seen physically based on the particular way how a quantum system with a certain number of degrees of freedom is decomposed into its subsystems as follows. While in the classical case one uses a Cartesian product

$$\mathbb{R}^{2k} \times \mathbb{R}^{2k} .. \times \mathbb{R}^{2k} = \mathbb{R}^{2kN}$$

to identify a decomposition into subsystems, one has to take into account for the quantum case a tensor product

$$\mathbb{C}^{k} \otimes \mathbb{C}^{k} .. \otimes \mathbb{C}^{k} \cong \mathbb{R}^{(2k)^N}.$$

Unitary operations defining a quantum dynamics may therefore act on a complex Hilbert space of dimension $k^N$.

The smallest quantum information unit is given by a so-called *qubit* being represented by a normalized vector in a 2-level complex Hilbert space isomorphic to $\mathbb{C}^2$ [10]. An initial state vector representing a circuit with 32 qubits for instance, allows therefore to consider unitary transformations inducing a *parallel quantum information processing* on a complex Hilbert space of more than four billion (!) dimensions.

Actually, the idea of quantum computation became of general interest a decade latter after Feynman's observation due to the work of Shor, proposing a quantum algorithm implying an exponential speed up for factorizing integers into prime numbers [11, 12].

Of course, solving 'classical' problems like prime number factorization on a quantum computer requires the extraction of classical information (that is a sequence of classical bits) from the final quantum state vector as output result of the corresponding quantum algorithm. This extraction is achieved by a measurement inducing 'a collapse' defined by a (non unitary) projection of the final quantum state vector

$$|\psi_{final}\rangle = U\,|\psi_{initial}\rangle = \sum_{j\in\{0,1\}^N}^{2^N} a_j\,|j\rangle \tag{1.1}$$

to a 1-dimensional subspace spanned by an eigenvector

$$|1011001...\rangle := |1\rangle \otimes |0\rangle \otimes |1\rangle \otimes |1\rangle \otimes |0\rangle \otimes |0\rangle \otimes |1\rangle \otimes ..$$

associated to an Hermitian operator on the composite Hilbert space

$$\mathcal{H}_{sys} \cong (\mathbb{C}^2)^{\otimes N}$$

representing the quantum register of $N$ qubits.

## 1.2 The role of entanglement

The final quantum state vector in the unitary process (1.1) will be in general *entagled* before the measurement occurs. Actually, any unitary quantum information process like (1.1) may be seen decomposed into a sequence of elementary unitary transformations provided by logical quantum gates inducing entanglement on each quantum state vector containing the information of the intermediate result related to the computational task. This may be illustrated in the simplest case of a CNOT (or 'controlled NOT') gate represented by a matrix

$$U_{CNOT} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{1.2}$$

acting on a quantum register consisting of two qubits according to

$$U_{CNOT}\,|x\rangle \otimes |y\rangle = |x\rangle \otimes |x \oplus y\rangle \tag{1.3}$$

with $x \oplus y := x + y \bmod 2$ (see e.g. [10]). Such an operation is *non-local* as the second qubit state changes in dependence of the first qubit state. The resulting state may therefore become entangled if the first qubit is in a superposition state, say

$$|x\rangle \equiv \sqrt{\lambda}\,|0\rangle + \sqrt{1-\lambda}\,|1\rangle\,, \quad \lambda \in [0,1].$$

As a matter of fact, for $|y\rangle \equiv |0\rangle$ one finds

$$\begin{aligned} |x\rangle \otimes |x \oplus y\rangle &= \sqrt{\lambda}\,|0\rangle \otimes |0 \oplus 0\rangle + \sqrt{1-\lambda}\,|1\rangle \otimes |0 \oplus 1\rangle \\ &= \sqrt{\lambda}\,|0\rangle \otimes |0\rangle + \sqrt{1-\lambda}\,|1\rangle \otimes |1\rangle \\ &= \sqrt{\lambda}\,|00\rangle + \sqrt{1-\lambda}\,|11\rangle \end{aligned} \tag{1.4}$$

which recovers a *Bell state* [13], that is, a *maximal entangled state* for $\lambda \equiv 1/2$. Indeed, any quantum algorithm may be seen decomposed into a finite sequence of *universal* quantum gates including the CNOT gate (1.3) generating entangled states as intermediate results in the corresponding quantum computational steps. Entanglement provides therefore a fundamental physical resource for realizing quantum computers.

## 1.3 Quantum error quantification

Unfortunately, there is a serious obstruction on the road to the concrete technological realizations of such quantum information processing devices. The main quantum engineering challenge arises here when taking precisely into account the interaction with the macroscopic environment. The latter implies in particular for quantum systems the destruction of the entanglement required for an error-free quantum information processing on the microscopic scale. Actually, this kind of destruction is a direct consequence of *entanglement with the environment* on the macroscopic scale (also known as 'decoherence'). An opposition between quantum information devices with optimal functionality and decoherence may therefore be captured by the notion of entanglement on different scales according to the following scheme.

Microscopic scale entanglement as resource for quantum error corrections [14]

vs.

Macroscopic scale entanglement as origin of errors on the microscopic scale.

This scheme makes clear that any serious quantum engineering approach to the realization and testing of quantum information devices requires a quantitative understanding on the relation between error tolerances and entanglement. Thus the elementary question which arises is how to quantify, both theoretically and experimentally, the error occurring in open quantum systems.

First of all, when taking into account an interaction with the environment, the unitary quantum information process becomes crucially modified to a non-unitary process, mathematically precisely defined by the notion of a (trace preserving) *positive map* $\Phi$ [15,16]. The latter encodes the unitarily evolved dynamics $U$ defining an interaction with the macroscopic environment down to the microscopic scale illustrated by the following diagram

$$\begin{tikzcd}
\mathcal{H}_{sys} \otimes \mathcal{H}_{env} \arrow[r, "U"] \arrow[d, "\text{`partial trace'}"'] & \mathcal{H}_{sys} \otimes \mathcal{H}_{env} \arrow[d, "\text{`partial trace'}"] \\
D(\mathcal{H}_{sys}) \arrow[r, "\Phi"] & D(\mathcal{H}_{sys}),
\end{tikzcd}$$

where $\mathcal{H}_{(\cdot)}$ denote Hilbert spaces associated with a subsystem ('sys' for the the microscopic system representing for instance a quantum computer register, and 'env' for the environment) and $D(\mathcal{H}_{sys})$ denotes the corresponding partial traced density state description of the microscopic subsystem.

To realize in this setting a quantum computer according to DiVincenzo [17] it becomes necessary to impose the constraint for any quantum state vector $|\psi\rangle \in \mathcal{H}_{sys}$ to differ only by a factor within an interval $[0, \epsilon]$ from a mixed state $\rho_\epsilon \in D(\mathcal{H}_{sys})$ after an elementary time unit according to

$$\langle\psi| \rho |\psi\rangle \geq 1 - \epsilon. \tag{1.5}$$

That is, a non-unitary process

$$|\psi\rangle \langle\psi| \mapsto \rho_\epsilon \equiv (1-\epsilon) |\psi\rangle \langle\psi| + \epsilon |\phi\rangle \langle\phi| \tag{1.6}$$

would be tolerable for performing quantum information processes with corresponding quantum error correction codes [14] if $\epsilon$ is small enough. The identification of the exact value of $\epsilon$ is therefore considered as one of the most crucial questions in the field of quantum computer engineering [17].

At this point we observe the following. Any measure on the space of quantum states $S : D(\mathcal{H}) \to [0, 1]$ being monotonic under positive maps would be able to give a quantification on the error. The dependence of a given measure $S$ on the error parameter $\epsilon$ may be induced here from the parametrization of a family of quantum states

$$\Phi_\epsilon(|\psi\rangle \langle\psi|) \equiv \rho_\epsilon, \tag{1.7}$$

each related to a fiducial state $\rho_0 \equiv |\psi\rangle \langle\psi|$ by virtue of a positive map $\Phi_\epsilon$ as illustrated in (1.6).

Actually, a complementary task in proving the functionality of quantum information processing devices could be given by the *quantification of entanglement*. Indeed, the quantification of errors *and* entanglement may turn out to appear both captured by one single measure known as the *von Neunman entropy*.

## 1.4 From quantum information to geometric QM

The traditional point of view in quantum information theory considers the von Neumann entropy

$$S_{vN}(\rho) := -\mathrm{Tr}(\rho \log \rho) \tag{1.8}$$

at the first place. It's fundamental role may be summarized by two remarkable facts: The von Neumann entropy provides

- the minimum amount of quantum information units needed for encoding a state $\rho$ (representing a typical sequence of eigenstates related to letters of an alphabet) without loosing information after decoding [18], and

- a unique entanglement measure for pure states, when applied to reduced density states [19].

At this point we may observe that the *quantum relative entropy*

$$S_{vN}(\rho,\sigma) \equiv S_{vN}(\rho) - \mathrm{Tr}(\rho \log \sigma), \tag{1.9}$$

has the interpretation of a 'distance' between quantum states which contracts under positive maps [16],

$$S(\Phi(\rho), \Phi(\sigma)) \leqslant S(\rho,\sigma). \tag{1.10}$$

Actually, the quantum relative entropy does *not* provide a *metrical* distance due to its lack of symmetry under permutation

$$S_{vN}(\rho,\sigma) \neq S_{vN}(\sigma,\rho). \tag{1.11}$$

However, it possible to associate a metrical distance with the Hessian

$$-\partial_\alpha \partial_\beta S_{vN}(\rho + \alpha A, \rho + \beta B)|_{\alpha=\beta=0} := M_\rho^{vN}(A,B) \tag{1.12}$$

defining a Riemmanian metric on the dense set of invertible mixed quantum state operators. Actually, there is a *family* of Riemmanian metrics constructed in this way [20], whenever taking into account *alternative* relative quantum entropies

$$S_h(\rho,\sigma) := \mathrm{Tr}(\rho\, h(\triangle_{\sigma,\rho})) \tag{1.13}$$

each defined in terms of an *operator convex function*[1] $h$, where $\triangle_{\sigma,\rho}$ denotes an operator being decomposed into a left and right action

$$\triangle_{\sigma,\rho}(A) := L_\sigma R_\rho^{-1}(A) = \sigma A \rho^{-1} \tag{1.14}$$

[1] We'll introduce this notion in more detail in section 3.1.1.

defined by the quantum states $\sigma$ and $\rho$ respectively.

To any given alternative quantum relative entropy $S_h$ specified by a operator convex function $h$, one finds a *quantum Fisher information metric*

$$-\partial_\alpha \partial_\beta S_h(\rho + \alpha A, \rho + \beta B)|_{\alpha=\beta=0} := M_\rho^h(A, B)$$

in accordance to the work of Petz, Gibilisco and al. (see e.g. [21] and references therein).

At this point we note the following. All quantum information metrics with an extension to pure states coincide with the *Fubini Study metric* when restricted to pure states. As being defined on the projective space of complex rays, it takes into account a fundamental structure in *the geometric formulation of quantum mechanics* [22–45].

## 1.5 An inverse approach: This thesis

The geometric formulation of quantum mechanics [22–45] has its historical origin in the pioneering works of Strocchi (1956), Cantoni (1975), Cirelli et al (1983). This approach is on the opposite to the *geometric quantization program* [46] by taking into account *dequantization* at the first place, as emphasized and worked out in recent developments by Ashtekar, Schilling, Brody, Hughston, Marmo et al. (1997-2010).

There are several sound reasons for considering a dequantization program in the geometric formulation of quantum mechanics. The initial motivation comes automatically by accepting physical states to be fundamentally realized as elements in the projective space of complex rays rather than in a ordinary Hilbert space. This space has a fundamental geometric structure provided by the Fubini-Study metric which decomposes along a realification into a real Riemannian and an imaginary symplectic structure. Such a geometric setting makes therefore available geometric methods as used in general relativity and classical mechanics, suggesting a powerful framework for approaching both the conceptual and mathematical foundations of quantum mechanics.

Actually, the specific relation of the Fubiny-Study metric to the quantum Fisher information, as outlined in the previous section, strongly suggests a particular impact on the mathematical foundations of quantum information. Indeed, we may ask more concretely for both conceptual and methodological implications of an inverse approach to quantum information theory, by setting the Fubiny-Study metric rather the von Neumann entropy at the first place.

A first step in this direction has been considered recently by a quantum entanglement characterization in finite level quantum systems [1–3] in terms of the pullback of the Fubini Study metric on local unitary group orbits of quantum states. Based on subsequent works [4–6], it is the aim of the present thesis to establish three further steps implying the following.

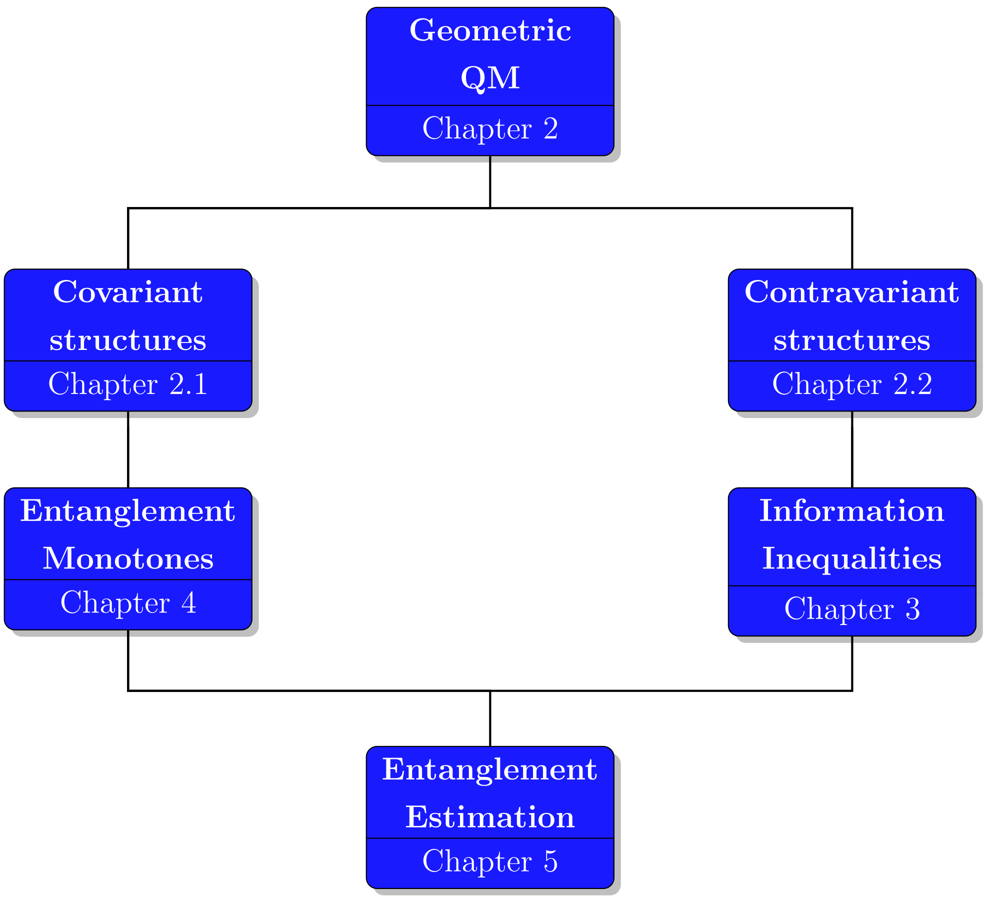
1. Information inequalities including the Cramér-Rao inequality from contravariant structures associated to the Fubini Study metric.

2. A (re-) construction of quantum entanglement monotones and related quantum entropies from the pull-back of the Fubini Study metric.

3. An efficient quantum estimation of weakly entangled qubits on the basis of the Cramér-Rao inequality in point 1 provided by an estimation of the monotones in point 2.

The last point 3 tackles a current raised problem on the experimental bounds of entanglement quantification [8, 47] and outlines therefore one of the possible advantages of an 'inverse approach', as proposed here, to quantum information.

The basic structure of the underlying work is illustrated according to the following diagram below. We'll start with the geometric formulation of quantum mechanics in section 2. This section will be focussed on the identification of covariant and contravariant structures related to the Fubini Study metric as seen from the Hilbert space in the subsections 2.1 and 2.2 respectively. Thereafter, we will consider alternative entanglement monotones arising from covariant structures in section 4 and quantum information inequalities arising from contravariant structures in section 3. In the last section 5 we'll bring both aspects together by illustrating an application to entanglement estimation.



## 2 Geometric Quantum Mechanics

The standard formulation of quantum mechanics associates to any quantum system a complex Hilbert space $\mathcal{H}$. However, on the basis of Born's probabilistic interpretation it becomes appropriated to define the physical states not with vectors $\psi$ in $\mathcal{H}$ but with the equivalence classes

$$[\psi] = [c\psi], \quad c \in \mathbb{C}_0 \tag{2.1}$$

which are elements of the *projective* Hilbert space $\mathcal{R}(\mathcal{H})$.[2] The projective Hilbert space carries a natural geometric structure given by the Fubini Study metric measuring the distance between two complex rays. The physically relevant distance between two quantum states should thus be considered in terms of the Fubini Study metric rather then in terms of a Hermitian scalar product. This observation highlights one of the motivations for considering a geometric formulation of quantum mechanics as suggested in several papers [22–45]. Here we shall review some of the basic ideas following the specific argumentaion line close to the reviews done in [2,5] with particular emphasize on the following notation:

To keep formulas both visible and computable with the familiar Dirac's 'ket'-notation it will be convinient to translate the physical relevant geometric information carried by the Fubini Study metric on $\mathcal{R}(\mathcal{H})$ on the level of the (punctured) Hilbert space $\mathcal{H}_0 \equiv \mathcal{H} - \{0\}$. This can be done both in terms of a covariant and a contravariant tensor field whenever $\mathcal{H}_0$ becomes identified with a differentiable manifold.

[2] At this point one may remark also alternative interpretations of quantum mechanics like the one by DeBroglie and Bohm, taking into account the quantum current density dependent quotient $j^\psi/|\psi|^2$ at the first place [48, 49]. The latter structure is invariant under $\mathbb{C}^*$-transformations on $\psi \in \mathcal{H}_0$ and provides therefore an alternative motivation for considering $\mathcal{R}(\mathcal{H})$ instead of $\mathcal{H}$ as the appropriated space of quantum states. Actually, a way of evading the measurement problem in such a setting may be seen based on the notion of *conditional* quantum states parametrized by the particle position configurations of the measurement device [50]. Such states may be modeled by a finite (macroscopic high) dimensional submanifold $Q_{apparatus} \subset \mathcal{R}(\mathcal{H})$.

One of the most appealing aspects in geometric quantum mechanics will be the translation of operator $C^*$-algebras to non-commutative $C^*$-algebras of functions. This translation makes use of a contravariant structure being related to the Fubini Study metric. In contrast, we will consider covariant structures when performing a pullback to general submanifolds in $\mathcal{H}$. It will therefore be convenient to distinguish covariant from contavariant structures by discussing them in separated subsections, as provided here in 2.1 and 2.2 respectively.

In what follows, our statements should be considered to be always mathematically well defined whenever the Hilbert space we intend to identify with a manifold is finite dimensional. Indeed, the basic ideas coming along the geometric approach in the finite dimensional case are fundamental for approaching the infinite dimensional case. The additional technicalities which may be required in the latter case will be discussed here by means of specific examples rather than by focusing on general claims. Readers interested in the mathematical foundations of infinite dimensional manifolds are invited to consult [51–53].

## 2.1 Covariant structures

Before considering the Fubiny Study metric in covariant form, we shall start with covariant tensor fields of lowest order.

### 2.1.1 From Hermitian operators to real-valued functions

Given a Hermitian operator $A \in u^*(\mathcal{H})$ defined on a Hilbert space $\mathcal{H}$, we shall find a real symmetric function

$$f_A(\psi) := \langle \psi \, | A\psi \rangle \,, \quad \psi \in \mathcal{H} \tag{2.2}$$

on $\mathcal{H}$. These functions decompose into functions

$$f_{P_j}(\psi) = \langle \psi \, | P_j \psi \rangle \,, \quad \psi \in \mathcal{H} \tag{2.3}$$

associated with a family of projectors $P_j := |e_j\rangle \langle e_j|$ constructed from an orthonormal basis $\{|e_j\rangle\}_{j \in I}$ on $\mathcal{H}$. The decomposition of the function induced by the spectral decomposition of the operator

$$A = \sum_j \lambda_j P_j \tag{2.4}$$

yields a *quadratic function*

$$f_A(\psi) = \sum_j \lambda_j f_{P_j}(\psi) = \sum_j \lambda_j |z^j|^2(\psi). \tag{2.5}$$

with the *coordinate functions*

$$\langle e_j \, |\psi\rangle := z^j(\psi) \,. \tag{2.6}$$

In this regard we may recover *expectation values of an operator $A$ as values of a function*

$$e_A(\psi) := \frac{f_A(\psi)}{\langle \psi \, |\psi\rangle} \tag{2.7}$$

on the punctured Hilbert space $\mathcal{H}_0 := \mathcal{H} - \{0\}$. By virtue of the map[3]

$$\mu : \mathcal{H}_0 \to u^*(\mathcal{H}), \quad |\psi\rangle \mapsto \rho_\psi := \frac{|\psi\rangle \langle \psi|}{\langle \psi \, |\psi\rangle} \tag{2.8}$$

[3]This map provides an instance of a so called *momentum map* as being known from Hamiltonian mechanics [54]. To be specific, for an action of a Lie group $\mathcal{G}$ on a symplectic manifold $(M, \omega)$ one defines a momentum map as a map $\mu$ from the symplectic manifold to the dual of the Lie algebra of $\mathcal{G}$, such that all $\mathbb{R}$-valued pairings between $\mu(v)$, $v \in M$ and elements $a$ of the Lie algebra generate a 1-form being the contraction of $\omega$ with a vector field associated with $a$ by $v \mapsto \frac{d}{dt} \exp(at)|_{t=0} v$. In the geometric formulation of quantum mechanics we may specialize this situation with $M \equiv \mathcal{H}_0$ and $\mathcal{G} \equiv U(\mathcal{H})$ [45]. The symplectic structure $\omega$ on $\mathcal{H}_0$ will be introduced later in section 2.2.

we note that

$$e_A(\psi) = \rho_\psi(A), \quad \rho_\psi \in D^1(\mathcal{H}) \tag{2.9}$$

identifies a pull-back function from the set $D^1(\mathcal{H})$ of normalized rank-1 projectors which are in 1-to-1 correspondence with pure physical states in $\mathcal{R}(\mathcal{H})$. Hence, $e_A$ is the result of the pull-back of a function form $\mathcal{R}(\mathcal{H})$ to $\mathcal{H}_0$.

### 2.1.2 The Fubini-Study metric seen from the Hilbert space

The map $\mu$ in (2.8) relates to the commutative diagram

$$\begin{array}{ccc} \mathcal{H}_0 & \xrightarrow{\mu} & u^*(\mathcal{H}) \\ {\scriptstyle\pi}\downarrow & & \uparrow{\scriptstyle\iota} \\ \mathcal{R}(\mathcal{H}) & \xrightarrow{\cong} & D^1(\mathcal{H}), \end{array}$$

providing a fundamental tool for pulling back, in a both computable and – with the familiar 'ket'-notation of Dirac – visible way, any covariant structure defined on $D^1(\mathcal{H}) \cong \mathcal{R}(\mathcal{H})$ to the 'initial' punctured Hilbert space $\mathcal{H}_0$.

In the following we will be particularly interested in the pullback of a covariant structure related to the Fubini Study metric on $\mathcal{R}(\mathcal{H})$. For this purpose, we consider for a given Hermitian operator $A$, the operator-valued differential $dA$ in respect to a real parametrization[4] of $u^*(\mathcal{H})$, and define the $(0, 2)$-tensor field

$$\mathrm{Tr}(dA \otimes dA). \tag{2.10}$$

The differential calculus on a submanifold $\mathcal{M} \subset u^*(\mathcal{H})$, may then be inherited from the 'ambient space' $u^*(\mathcal{H})$ together with this covariant structure. In

[4]Such a parametrization is available, for instance in terms of the Bloch representation $A = \sum \lambda_j \sigma^j$ with $\lambda_j := \mathrm{Tr}(A\sigma^j)$. An explicit computation of the resulting tensor field is illustrated for $\mathcal{H} \cong \mathbb{C}^2$ in the appendix section B.4.

particular for $\mathcal{M} \cong D^1(\mathcal{H}) \cong \mathcal{R}(\mathcal{H})$ we find[5]

$$\mathrm{Tr}(d\rho_\psi \otimes d\rho_\psi) = \frac{\langle d\psi \otimes d\psi\rangle}{\langle \psi\, |\psi\rangle} - \frac{\langle \psi\, |d\psi\rangle}{\langle \psi\, |\psi\rangle} \otimes \frac{\langle d\psi\, |\psi\rangle}{\langle \psi\, |\psi\rangle} := \kappa_{\mathcal{H}_0}\,, \tag{2.11}$$

as $\mu$-map (2.8) induced pull-back tensor field on the associated punctured Hilbert space $\mathcal{H}_0$ [1]. We note here that $|d\psi\rangle$ defines a $\mathcal{H}$-vector-valued 1-form which provides a 'classical' 1-form according to $\langle e_j\, |d\psi\rangle \equiv dz^j$. In the latter coordinates one may identify a degenerate covariant tensor field

$$\kappa_{\mathcal{H}_0} = \frac{d\bar{z}^j \otimes dz^j}{\sum_j |z^j|^2} - \frac{z^j d\bar{z}^j \otimes \bar{z}^k dz^k}{(\sum_j |z^j|^2)^2} \tag{2.12}$$

on the punctured Hilbert space $\mathcal{H}_0 \cong \mathbb{C}^{n+1} - \{0\}$. Due to the misleading convention often appearing in the literature we shall give a warning at this point: This is *not* the Fubini Study metric from the associated projective space $\mathbb{C}P^n$. Indeed, the above covariant tensor-field $\kappa_{\mathcal{H}_0}$ defines *a pull-back of the Fubini-Study metric tensor field from the space of complexe rays $\mathcal{R}(\mathcal{H}) \cong \mathbb{C}P^n$ to $\mathcal{H}_0 \cong \mathbb{C}^{n+1} - \{0\}$*.

The degenerate structure $\kappa_{\mathcal{H}_0}$ (2.11) decomposes in this regard into a real symmetric and an imaginary anti-symmetric part

$$\kappa_{\mathcal{H}_0} := \eta_{\mathcal{H}_0} + i\omega_{\mathcal{H}_0}, \tag{2.13}$$

relating to a corresponding *pullback of a Riemannian and a symplectic structures from the associated complex projective space to $\mathcal{H}_0$* respectively. Of course, such a decomposition may either be induced by polar coordinates $z^j = p^j e^{iW^j}$ or by cartesian coordinates $z^j \equiv x^j + iy^j$.

[5] A detailed derivation of formula (2.11) can be found in the appendix C in (C.8).

### 2.1.3 The pullback on general submanifolds

For a given embedding of a general finite dimensional manifold $\mathcal{M}$ in $\mathcal{H}_0$

$$\iota : \mathcal{M} \hookrightarrow \mathcal{H}_0, \quad \lambda \mapsto \psi(\cdot, \lambda) \tag{2.14}$$

we will find an induced pullback of the Fubini-Study metric on $\mathcal{M}$ in terms of the pullback of the degenerate covariant structure given by $\kappa_{\mathcal{H}_0}$ in (2.11).

Let us review an explicit derivation of the pullback when the Hilbert space under consideration is identified with a space of square integrable functions on some configuration space $\mathbb{R}^n$ [7]. As a first step we consider here a polar coordinate decomposition

$$\psi(x, \lambda) \equiv p(x, \lambda)^{1/2} e^{iW(x,\lambda)} \tag{2.15}$$

and define for any given tensor field $T(x, \lambda)$ of order $r$ (including functions for order $r = 0$) the generalized expectation value integral

$$\mathbb{E}_p(T) := \int_{\mathbb{R}^n} p(x, \lambda) T(x, \lambda) dx, \tag{2.16}$$

which 'traces out' the $x$-dependence of the tensor field $T$. Hence, for a given embedding (2.14) we shall find the pull-back structures

$$\iota^* \langle \psi \, | \psi \rangle = \mathbb{E}_p(1) \tag{2.17}$$

$$\iota^* \langle \psi \, | d\psi \rangle = \mathbb{E}_p(d \ln \psi) \tag{2.18}$$

$$\iota^* \langle d\psi \, | \psi \rangle = \mathbb{E}_p(d \ln \psi^*) \tag{2.19}$$

$$\iota^* \langle d\psi \otimes d\psi \rangle = \mathbb{E}_p(d \ln \psi^* \otimes d \ln \psi), \tag{2.20}$$

by using

$$\frac{d\psi}{\psi} = d \ln \psi. \tag{2.21}$$

The latter splits within the polar-decompostion (2.15) into a sum

$$d\ln\psi = d(\ln p^{1/2}e^{iW}) = d(\ln p^{1/2} + \ln(e^{iW})) = \frac{1}{2}d\ln p + idW. \tag{2.22}$$

By taking into account the normalization condition $\langle\psi\,|\psi\rangle = 1$ one finds

$$d\mathbb{E}_p(1) = \mathbb{E}_p(dp) = \mathbb{E}_p(d\ln p) = 0 \tag{2.23}$$

and $\langle d\psi\,|\psi\rangle = -\,\langle\psi\,|d\psi\rangle$,

$$\mathbb{E}_p(d\ln\psi^*) = -\mathbb{E}_p(d\ln\psi). \tag{2.24}$$

From (2.16)-(2.24) we conclude the identification of a pull-back tensor field

$$\iota^*\kappa_{\mathcal{H}_0} \equiv \kappa_{\mathcal{M}} = \eta_{\mathcal{M}} + i\omega_{\mathcal{M}}, \tag{2.25}$$

on the submanifold $\mathcal{M}$ which is decomposed into a symmetric tensor field

$$\eta_{\mathcal{M}} := \mathbb{E}_p((d\ln p)^{\otimes 2}) + \mathbb{E}_p(dW^{\otimes 2}) - \mathbb{E}_p(dW)^2 \tag{2.26}$$

and an antisymmetric tensor field

$$\omega_{\mathcal{M}} := \mathbb{E}_p(d\ln p \wedge dW). \tag{2.27}$$

While the latter is related to the geometric phase, we shall take into account in the symmetric part a further decomposition

$$\eta_{\mathcal{M}} \equiv \mathcal{F} + \mathrm{Cov}(dW), \tag{2.28}$$

which identifies the *classical Fisher Information metric*

$$\mathcal{F} := \mathbb{E}_p((d\ln p)^{\otimes 2}) \tag{2.29}$$

and a *phase-covariance matrix tensor field*

$$\mathrm{Cov}(dW) := \mathbb{E}_p(dW^{\otimes 2}) - \mathbb{E}_p(dW)^2. \tag{2.30}$$

For the parts of the pull-back tensor field containing the phase $W$ in differential form, we may therefore identify for pure states according to [7] the non-classical counterpart of the Fisher classical within the quantum information metric. As a matter of fact, the quantum information metric collapses to the *classical Fisher information metric* for

$$dW = 0. \tag{2.31}$$

### 2.1.4 An example

As a possible application we may show in the following the existence of a non-trivial Ricci curvature of a submanifold of quantum states. Consider for this purpose a family of two-modes coherent state vectors

$$|\psi_{q,p}\rangle := |q,p\rangle \in L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \cong L^2(\mathbb{R}^2) \tag{2.32}$$

parametrized by four-dimensional phase space vectors

$$(q,p) := (q_1, q_2, p_1, p_2) \in T^*(\mathbb{R} \times \mathbb{R}), \tag{2.33}$$

which induce a parametrization of position representation wave functions

$$\langle x_1, x_2 \,|q,p\rangle = \psi_{q,p}(x_1,x_2) := \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(q_1+x_1)^2-(q_2+x_2)^2}{2\sigma^2}} e^{ip_1x_1+ip_1x_2} \tag{2.34}$$

with an overall width $\sigma \in \mathbb{R}_+$. At this point we may consider the latter as an additional variable parameter, and describe a 5-dimensional submanifold of quantum state vectors

$$\psi_{q,p,\sigma}(x_1, x_2) := \psi(x, \lambda) \equiv \frac{1}{\sqrt{2\pi\lambda_5^2}} e^{\frac{-(\lambda_1+x_1)^2-(\lambda_2+x_2)^2}{2\lambda_5}} e^{i\lambda_3 x_1 + i\lambda_4 x_2} \tag{2.35}$$

parametrized by

$$(q_1, q_2, p_1, p_2, \sigma) \equiv (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \equiv \lambda \in \mathbb{R}^4 \times \mathbb{R}_+. \tag{2.36}$$

Using the polar decomposition $\psi(x, \lambda) \equiv p(x, \lambda)^{1/2} e^{W(x,\lambda)}$ as described in (2.15) we identify the modulo of the wave function

$$p(x, \lambda) = \frac{1}{2\pi\lambda_5^2} e^{-\frac{(x_1+\lambda_1)^2+(x_2+\lambda_2)^2}{\lambda_5^2}} \tag{2.37}$$

and the phase

$$W(x, \lambda) = x_1\lambda_3 + x_2\lambda_4. \tag{2.38}$$

Using the formulas (2.26)-(2.30) we find a degenerate symmetric tensor coefficient matrix

$$(\mathcal{F}_{jk}) = \left(\mathbb{E}_p\left(\frac{\partial \ln p}{\partial \lambda_j}\frac{\partial \ln p}{\partial \lambda_k}\right)\right) = \begin{pmatrix} \frac{1}{\lambda_5^2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\lambda_5^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{\lambda_5^2} \end{pmatrix} \tag{2.39}$$

being related to the classical Fischer information metric on a 3-dimensional submanifold, and a phase-covariance matrix tensor

$$\mathrm{Cov}\left(\frac{\partial W}{\partial \lambda_j}, \frac{\partial W}{\partial \lambda_k}\right) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4}\left(2\lambda_1^2+\lambda_5^2\right)-\frac{\lambda_1^2}{4} & \frac{\lambda_1\lambda_2}{4} & 0 \\ 0 & 0 & \frac{\lambda_1\lambda_2}{4} & \frac{1}{4}\left(2\lambda_2^2+\lambda_5^2\right)-\frac{\lambda_2^2}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.40}$$

The symmetric part of the pull-back $\kappa_{\mathcal{M}}$ of (2.11) implies therefore the sum of the two tensor fields, which yield the 'quantum' Riemannian structure

$$\eta_{\mathcal{M}} = \begin{pmatrix} \frac{1}{\lambda_5^2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\lambda_5^2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4}\left(2\lambda_1^2+\lambda_5^2\right)-\frac{\lambda_1^2}{4} & \frac{\lambda_1\lambda_2}{4} & 0 \\ 0 & 0 & \frac{\lambda_1\lambda_2}{4} & \frac{1}{4}\left(2\lambda_2^2+\lambda_5^2\right)-\frac{\lambda_2^2}{4} & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{\lambda_5^2} \end{pmatrix}. \tag{2.41}$$

Based on this Riemannian metric defined on a 5-dimensional submanifold of coherent states, we may compute the Ricci scalar curvature

$$R = (\eta_{jk})^{-1} R_{jk} = -\frac{5\lambda_1^4 + 5\left(2\lambda_2^2+3\lambda_5^2\right)\lambda_1^2 + 5\lambda_2^4 + 12\lambda_5^4 + 15\lambda_2^2\lambda_5^2}{2\left(\lambda_1^2+\lambda_2^2+\lambda_5^2\right)^2} \tag{2.42}$$

from the result of the associated Riemannian curvature tensor $R^k_{ijl}$ as illustrateted on the next page.

$$(R_{jk}) = (R^k_{jkj}) = \begin{pmatrix} -\frac{\lambda_1^4+\left(3\lambda_2^2+\lambda_5^2\right)\lambda_1^2+2\left(\lambda_2^2+\lambda_5^2\right)^2}{2\lambda_5^2\left(\lambda_1^2+\lambda_2^2+\lambda_5^2\right)^2} & \frac{\lambda_1\lambda_2\left(\lambda_1^2+\lambda_2^2+3\lambda_5^2\right)}{2\lambda_5^2\left(\lambda_1^2+\lambda_2^2+\lambda_5^2\right)^2} & 0 & 0 & -\frac{\lambda_1\left(\lambda_1^2+\lambda_2^2\right)}{\lambda_5\left(\lambda_1^2+\lambda_2^2+\lambda_5^2\right)^2} \\ \frac{\lambda_1\lambda_2\left(\lambda_1^2+\lambda_2^2+3\lambda_5^2\right)}{2\lambda_5^2\left(\lambda_1^2+\lambda_2^2+\lambda_5^2\right)^2} & -\frac{2\lambda_1^4+3\lambda_2^2\lambda_1^2+\lambda_2^4+2\lambda_5^4+\left(4\lambda_1^2+\lambda_2^2\right)\lambda_5^2}{2\lambda_5^2\left(\lambda_1^2+\lambda_2^2+\lambda_5^2\right)^2} & 0 & 0 & -\frac{\lambda_2\left(\lambda_1^2+\lambda_2^2\right)}{\lambda_5\left(\lambda_1^2+\lambda_2^2+\lambda_5^2\right)^2} \\ 0 & 0 & \frac{\lambda_1^4+\lambda_2^2\lambda_1^2-2\lambda_5^4}{8\left(\lambda_1^2+\lambda_2^2+\lambda_5^2\right)} & \frac{\lambda_1\lambda_2\left(\lambda_1^2+\lambda_2^2\right)}{8\left(\lambda_1^2+\lambda_2^2+\lambda_5^2\right)} & 0 \\ 0 & 0 & \frac{\lambda_1\lambda_2\left(\lambda_1^2+\lambda_2^2\right)}{8\left(\lambda_1^2+\lambda_2^2+\lambda_5^2\right)} & \frac{\lambda_2^4+\lambda_1^2\lambda_2^2-2\lambda_5^4}{8\left(\lambda_1^2+\lambda_2^2+\lambda_5^2\right)} & 0 \\ -\frac{\lambda_1\left(\lambda_1^2+\lambda_2^2\right)}{\lambda_5\left(\lambda_1^2+\lambda_2^2+\lambda_5^2\right)^2} & -\frac{\lambda_2\left(\lambda_1^2+\lambda_2^2\right)}{\lambda_5\left(\lambda_1^2+\lambda_2^2+\lambda_5^2\right)^2} & 0 & 0 & -\frac{\left(\lambda_1^2+\lambda_2^2+2\lambda_5^2\right)\left(2\lambda_5^2+3\left(\lambda_1^2+\lambda_2^2\right)\right)}{\lambda_5^2\left(\lambda_1^2+\lambda_2^2+\lambda_5^2\right)^2} \end{pmatrix}$$

### 2.1.5 The pull-back on homogenous spaces

Hilbert spaces being relevant in the description of a quantum mechanical system may be decomposed in terms of irreducible unitary representations of a Lie Group $\mathcal{G}$. This implies that any covariant structure defined on a Hilbert space will admit a pull-back to a homogenous space manifold of quantum states in dependence of the chosen Lie group action. In particular for the Fubini Study metric related degenerate covariant structure $\kappa_{\mathcal{H}_0}$ in (2.11) one finds the following statement [1, 3, 55].

**Theorem 2.1.** *Let $\{\theta_j\}_{j\in J}$ be a basis of left-invariant 1-forms on a Lie group $\mathcal{G}$, and let $\{X_j\}_{j\in J}$ be a dual basis of left-invariant vector fields, and let $iR$ be the Lie algebra representation associated to the unitary representation $U : \mathcal{G} \to U(\mathcal{H})$, inducing by means of any fiducial state vector $|\psi\rangle \in S(\mathcal{H})$ a map*

$$\iota_{\mathcal{G}} : \mathcal{G} \to \mathcal{H},$$

$$\iota_{\mathcal{G}}(g) := U(g)\,|\psi\rangle\,.$$

*Then*

$$\iota_{\mathcal{G}}^{*}\kappa_{\mathcal{H}_0} = Cov_{\rho_\psi}((R(X_j)R(X_k))\theta^j \otimes \theta^k := \kappa_{\mathcal{G}}^{\rho_\psi} \tag{2.43}$$

*for $\rho_\psi := |\psi\rangle\langle\psi| \in D^1(\mathcal{H})$.*

In conclusion, the pull-back of $\kappa_{\mathcal{H}_0}$ on a Lie group endowed with a unitary Hilbert space representation gives rise to a *covariance tensor field*

$$(\rho_\psi(R(X_j)R(X_k)) - \rho_\psi(R(X_j))\rho_\psi(R(X_k))\theta^j \otimes \theta^k. \tag{2.44}$$

which reduces in particular for 1-dimensional representations to a *variance tensor field*

$$Var_{\rho_\psi}(R(X))\theta \otimes \theta. \tag{2.45}$$

We may identify this (degenerate) pull-back tensor field construction with the pull-back of a non-degenerate pull-back tensor field which lives on a homogenous space $\mathcal{G}/\mathcal{G}_0$. The latter admits a smooth embedding via the unitary action of the Lie Group as orbit manifold $\Gamma$ in the Hilbert space and establishes therefore a pull-back of the Hermitian structure both on the orbit $\Gamma$ and the homogenous space $\mathcal{G}/\mathcal{G}_0$. Hence, the computation of the pull-back on the orbit, reduces to the the computation of the pull-back on the Lie group, as indicated here in the commutative diagram below.

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\iota_{\mathcal{G}}} & \mathcal{H} \\ {\scriptstyle\pi}\Big\downarrow & & \Big\uparrow{\scriptstyle\iota} \\ \mathcal{G}/\mathcal{G}_0 & \xrightarrow{\cong} & \Gamma. \end{array}$$

The embedding of the Lie group and its corresponding orbit is related to the co-adjoint action map on all group elements modulo U(1)-representations $U(h) = e^{i\phi(h)}$

$$\iota_{\mathcal{G}}^{U(1)} : \mathcal{G}/U(1) \to \mathcal{R}(\mathcal{H}), \quad g \mapsto U(g)\rho U(g)^{\dagger}, \quad \rho \in \mathcal{R}(\mathcal{H}). \tag{2.46}$$

Let us underline again: The structure (2.44) is defined *on the Lie group via a pull-back tensor field from the Hilbert space* even though it contains the complete information of the (non-degenerate) tensor field on the corresponding co-adjoint orbit $\Gamma$ which is embedded in the projective Hilbert space. The additional $U(1)$- degeneracy is here captured in a corresponding enlarged isotropy group $\mathcal{G}_0^{U(1)}$ according the commutative diagram below.

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\iota_{\mathcal{G}}} & S(\mathcal{H}) \\ {\scriptstyle U(1)}\Big\downarrow & & \Big\downarrow{\scriptstyle U(1)} \\ \mathcal{G}/U(1) & \xrightarrow{\iota_{\mathcal{G}}^{U(1)}} & \mathcal{R}(\mathcal{H}) \\ {\scriptstyle\pi}\Big\downarrow & & \Big\uparrow{\scriptstyle\iota} \\ \mathcal{G}/\mathcal{G}_0^{U(1)} & \xrightarrow{\cong} & \Gamma \end{array}$$

This approach provides therefore in an 'algorithmic' procedure to find a geometric description of coherent state manifolds, as defined in [56–58]. Indeed, the associated orbits in our approach turn out to be more general as those give by coherent states, whenever we allow to take into account also reducible representations, as it typically occurs in composite Hilbert spaces. We'll illustrate this later on in section 4.1.

### 2.1.6 Example: The pullback in a Weyl-system

Let $\{X_j\}_{j\in J}$ be a basis on $V \cong \mathbb{R}^{2n}$ represented within a Weyl system according to the definition in section A.1 by a set of Hermitian operators $\{R(X_j)\}_{j\in J}$ on $\mathcal{H} \cong L^2(U)$, where $U \cong \mathbb{R}^n$ defines a Lagrangian subspace of $V$. Given a pure state $\rho_\psi = |\psi\rangle\langle\psi|$ associated to a normalized vector $\psi \in S(\mathcal{H})$, we identify

$$Cov_{\rho_\psi}((R(X_j)R(X_k))dv^j \otimes dv^k \tag{2.47}$$

as the pull-back tensor field of the covariant tensor field $\kappa_{\mathcal{H}_0}$ as defined in (2.11) from $\mathcal{H}_0$ to $V$. This pullback is induced by a map

$$V \to \mathcal{H}, \quad v \mapsto W(v)\,|\psi\rangle \tag{2.48}$$

specified by the Weyl-system map $W : V \to U(\mathcal{H})$ with $W(v) = e^{iR(v)}$ (see appendix A) and the choice of a fiducial state vector $\psi \in S(\mathcal{H})$. This follows from a generalization of theorem 2.1 to infinite dimensional Hilbert spaces, once we impose the additional condition that the fiducial vector $\psi$ is smooth and in the domain of the Hermitian operators $R(X_j)$ [55]. While the anti-symmetric coefficients recover the symplectic structure[6]

$$\rho_\psi([R(X_j)R(X_k)]) = \frac{i}{4a^2}\omega_{[jk]}, \tag{2.49}$$

due to

$$[R(v)R(v')] = \frac{i}{2a^2}\omega(v,v') \text{ and } \mathrm{Tr}(\rho_\psi) = 1, \tag{2.50}$$

*independently* from the state $\rho_\psi$, its symmetric tensor coefficients are identical to the coefficients of a symmetric covariance matrix $\sigma$ *in dependence of the state*:

$$\rho_\psi([R(X_j)R(X_k)]_+) - \rho_\psi(R(X_j))\rho_\psi(R(X_k)) := \sigma_{jk}(\rho_\psi). \tag{2.51}$$

In conclusion, the Weyl-system-induced pull-back of $\kappa_{\mathcal{H}_0}$ in (2.11) establishes by its tensor coefficient matrix a $n \times n$ covariance matrix

$$\{\mathrm{Cov}_{\rho_\psi}(R(X_j), R(X_k)\}_{j,k}, \tag{2.52}$$

being decomposed into a quantum state-dependent real symmetric part and a quantum state-independent imaginary anti-symmetric part

$$\sigma(\rho_\psi) + \frac{i}{4a^2}\omega. \tag{2.53}$$

The anti-symmetric part $\omega$ stays invariant by definition under symplectic transformations. In contrast, we shall encounter in the symmetric part $\sigma(\rho_\psi)$ a non-trivial transformation. In particular, by the virtue of *Williamson's theorem* [60], there exists to any real symmetric matrix $\sigma \in M_{2n}(\mathbb{R})$ a symplectic

[6]We adopt the convention as used in [59] for the commutation relations involving

$$a := \begin{cases} 2^{-1/2} \text{ for canonical position and momentum} \\ 1 \text{ for optical position and momentum.} \end{cases}$$

transformation matrix $S \in Sp(2n, \mathbb{R})$ such that

$$S\sigma S^T = \bigoplus_{k=1}^{n} d_k \mathbb{1}_2, \quad d_k \in \mathrm{spec}(i\omega \cdot \sigma) \tag{2.54}$$

where $d_k$ denote the *symplectic eigenvalues* of the matrix product $i\omega \cdot \sigma$.

## 2.2 Contravariant structures

So far we considered covariant structures derived as pullback tensor fields of a degenerate covariant tensor field $\kappa_{\mathcal{H}_0}$. The latter has been related to a pullback of the Fubini Study metric from the projective Hilbert space to the punctured Hilbert space.

At this point we note, it is not possible to associate to a degeneratate covariant structure $(\kappa_{jk})$ a corresponding *contravariant* structure $(\kappa^{jk}) = (\kappa_{jk})^{-1}$. However, it is possible to define contravariant structures on the Hilbert space which are projectable on the space of complex rays. This is done in two steps. First by considering a Hermitian structure turning a Hilbert space to a Hilbert manifold, and second, by extending this Hermitian structure in terms of dilatation and phase generating vector fields to a projectable tensor field. A benefit of such a procedure is the possiblity to translate Hilbert space operator products to star products of functions on unitary orbits of pure quantum states.

### 2.2.1 From Hilbert spaces to Hilbert manifolds

By introducing an orthonormal basis $\{|e_j\rangle\}_{j\in J}$ on a given Hilbert space $\mathcal{H}$, we may define coordinate functions by setting

$$\langle e_j \,|\psi\rangle = z^j(\psi), \tag{2.55}$$

which we'll write in the following simply as $z^j$. Correspondently, for the dual basis $\{\langle e_j|\}$ we find coordinate functions

$$\langle \psi \,|e_j\rangle = \bar{z}_j(\psi^*) \tag{2.56}$$

defined on the dual space $\mathcal{H}^*$. By using the inner product we can identify $\mathcal{H}$ and $\mathcal{H}^*$. This provides two possibilities: The scalar product $\langle \psi \,|\psi\rangle$ gives rise to a covariant Hermitian (0, 2)-metric tensor on $\mathcal{H}$

$$\langle d\psi \,| d\psi\rangle = \sum_j \langle d\psi \,| e_j\rangle \langle e_j \,| d\psi\rangle = d\bar{z}_j \otimes dz^j, \tag{2.57}$$

where we have used $d\langle e_j \,|\psi\rangle = \langle e_j \,| d\psi\rangle$, i.e., the chosen basis is not 'varied', or to a contra-variant (2,0) tensor

$$\left\langle \frac{\partial}{\partial\psi} \,\middle|\, \frac{\partial}{\partial\psi} \right\rangle = \frac{\partial}{\partial\bar{z}_j} \otimes \frac{\partial}{\partial z^j} \tag{2.58}$$

on $\mathcal{H}^*$.

*Remark:* Specifically, we assume that an orthonormal basis has been selected once and it does not depend on the base point.

By introducing real coordinates, say

$$z^j(\psi) = x^j(\psi) + iy^j(\psi) \tag{2.59}$$

one finds

$$\langle d\psi \,| d\psi\rangle = (dx_j \otimes dx^j + dy_j \otimes dy^j) + i(dx_j \otimes dy^j - dy_j \otimes dx^j). \tag{2.60}$$

Thus the Hermitian tensor decomposes into an Euclidean metric (more generally a Riemannian tensor) and a symplectic form.
Similarly, on $\mathcal{H}^*$ we may consider

$$\left\langle \frac{\partial}{\partial\psi} \,\middle|\, \frac{\partial}{\partial\psi} \right\rangle = \left( \frac{\partial}{\partial x_j} \otimes \frac{\partial}{\partial x^j} + \frac{\partial}{\partial y_j} \otimes \frac{\partial}{\partial y^j} \right) + i \left( \frac{\partial}{\partial y_j} \otimes \frac{\partial}{\partial x^j} - \frac{\partial}{\partial x_j} \otimes \frac{\partial}{\partial y^j} \right).$$

This tensor field, in contravariant form, may be also considered as a bi-differential operator, i.e., we may define a binary bilinear product on real smooth functions by setting

$$((f, g)) = \left(\frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j}\right) \cdot \left(\frac{\partial g}{\partial x^j} - i\frac{\partial g}{\partial y^j}\right) \tag{2.61}$$

which decomposes into a symmetric bracket

$$(f, g) = \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x^j} + \frac{\partial f}{\partial y_j}\frac{\partial g}{\partial y^j} \tag{2.62}$$

and a skew-symmetric bracket

$$\{f, g\} = \frac{\partial f}{\partial y_j}\frac{\partial g}{\partial x^j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial y^j}. \tag{2.63}$$

This last bracket defines a Poisson bracket on smooth functions defined on $\mathcal{H}$.

Summarizing, we can replace our original Hilbert space with an Hilbert manifold, i.e. an even dimensional real manifold on which we have tensor fields in covariant form[7]

$$\eta_{\mathcal{H}} = dx_j \otimes dx^j + dy_j \otimes dy^j \tag{2.64}$$

$$\omega_{\mathcal{H}} = dy_j \otimes dx^j - dx_j \otimes dy^j, \tag{2.65}$$

or tensor fields in contravariant form

$$G_{\mathcal{H}} = \frac{\partial}{\partial x_j} \otimes \frac{\partial}{\partial x^j} + \frac{\partial}{\partial y_j} \otimes \frac{\partial}{\partial y^j} \tag{2.66}$$

$$\Omega_{\mathcal{H}} = \frac{\partial}{\partial y_j} \otimes \frac{\partial}{\partial x^j} - \frac{\partial}{\partial x_j} \otimes \frac{\partial}{\partial y^j} \tag{2.67}$$

defining covariant and contravariant Riemannian and symplectic structures respectively. The contravariant tensor fields, considered as bi-differential

[7]We shall distinguish these tensor from the symmetric and antisymmetric part of the pullback tensor $\kappa_{\mathcal{H}_0}$, as being related in (2.11) to the Fubini Study metric, by leaving out the index '0' within the Hilbert space notation.

operators define a symmetric product and a skew symmetric product on real smooth functions. The skew-symmetric product actually defines a Poisson bracket. In particular, for functions

$$f_A(\psi) = \langle \psi \, | A\psi \rangle \, , \quad \psi \in \mathcal{H}, \tag{2.68}$$

associated with Hermitian operators $A$, we shall end up with the relations

$$f_{[A,B]_+} \equiv G_{\mathcal{H}}(df_A, df_B). \tag{2.69}$$

$$f_{[A,B]_-} \equiv \Omega_{\mathcal{H}}(df_A, df_B), \tag{2.70}$$

which replaces symmetric and anti-symmetric operator products $[A, B]_\pm$ by symmetric and anti-symmetric tensor fields respectively. Hence, via these tensor fields we may identify symmetric and Poisson brackets on the set of quadratic functions according to

$$f_{[A,B]_+} = (f_A, f_B), \tag{2.71}$$

$$f_{[A,B]_-} = \{f_A, f_B\}, \tag{2.72}$$

which synthesize to a non-commutative product

$$((f_A, f_B)) = (f_A, f_B) + i\{f_A, f_B\} \tag{2.73}$$

of quadratic functions. In this way we may encode the original non-commutative structure on operators in terms of 'classical', i.e. Riemannian and symplectic tensor fields according to

$$((f_A, f_B)) = f_{A \cdot B}(\psi) = (G_{\mathcal{H}} + i\Omega_{\mathcal{H}})(df_A(\psi), df_B(\psi)). \tag{2.74}$$

### 2.2.2 Projectable tensor fields

To take into account the geometry of the set of physical (pure) states, we need to modify $G$ and $\Omega$ by a conformal factor to turn them into projectable tensor fields on $\mathcal{R}(\mathcal{H})$. The projection is generated at the infinitesimal level by the real and imaginary parts of the action of $\mathbb{C}_0$ on $\mathcal{H}_0$ given by the dilation vector field $\Delta$ and the $U(1)$-phase rotation generating vector field $\Gamma := J(\Delta)$ defined by a contraction with a complex structure (1-1)-tensor field

$$J_{\mathcal{H}} = dx^j \otimes \frac{\partial}{\partial y_j} - dy^j \otimes \frac{\partial}{\partial x^j} \tag{2.75}$$

respectively. In this way we shall identify

$$\widetilde{G}(\psi) = \langle \psi \, | \psi \rangle \, G - (\Delta \otimes \Delta + \Gamma \otimes \Gamma) \tag{2.76}$$

$$\widetilde{\Omega}(\psi) = \langle \psi \, | \psi \rangle \, \Omega - (\Delta \otimes \Gamma - \Gamma \otimes \Delta), \tag{2.77}$$

as projectable structures [61]. They establish a Lie-Jordan algebra structure on the space of real valued functions whose Hamiltonian vector fields are also Killing vector fields for the projection $\tilde{G}$. In this regard one finds: A function on $\mathcal{R}(\mathcal{H})$ defines a quantum evolution, via the associated Hamiltonian vector field, if and only if the vector field is a derivation for the Riemann-Jordan product [39, 62].

### 2.2.3 Non-commutative $C^*$-algebras of Kähler functions

For a given (bounded) operator $A \in \mathcal{B}(\mathcal{H})$ on a Hilbert space we consider the complex valued function $f_A : \mathcal{H}_0 \to \mathbb{C}$ on the punctured Hilbert space defined by

$$f_A(\psi) \equiv \frac{\langle \psi | \, A \, | \psi \rangle}{\langle \psi \, | \psi \rangle} \tag{2.78}$$

with $\psi \in \mathcal{H}_0 := \mathcal{H} - \{0\}$. By introducing a contravariant tensor field $K_{\mathcal{H}_0}$ on $\mathcal{H}_0$ being projectable on the space of complex rays $\mathcal{R}(\mathcal{H})$ we shall find a decomposition of the function $f_{AB}$ on a product of two operators $A$ and $B$ into

$$f_{AB}(\psi) = f_A \cdot f_B|_\psi + K_{\mathcal{H}_0}(df_A, df_B)|_\psi. \tag{2.79}$$

That is, a point-wise product and a contraction of the contra-variant Hermitian tensor field $K_{\mathcal{H}_0}$ with the differential 1-forms $df_A$ and $df_B$ yielding a non-commutative star-product of functions

$$f_A \cdot f_B|_\psi + K_{\mathcal{H}_0}(df_A, df_B)|_\psi := f_A \star f_B|_\psi. \tag{2.80}$$

This product defines a $C^*$-algebra for all functions on $\mathcal{R}(\mathcal{H})$ provided by (2.78). These functions turn out to be Kähler function as they generate in their real and imaginary part Hamiltonian vector fields which are also Killing vector fields [40]. Note that the corresponding pullback of Kähler functions on a submanifold $\Gamma \subset \mathcal{R}(\mathcal{H})$ yields again a $C^*$-algebra iff $\Gamma$ is a unitarily generated homogenous space [61]. In particular, this includes also $\mathcal{R}(\mathcal{H})$ as an instance of a unitary orbit in the case of finite dimensional Hilbert spaces via the bijection

$$\mathcal{R}(\mathcal{H}) \leftrightarrow U(N)/U(N-1) \times U(1), \quad N = \mathrm{Dim}(\mathcal{H}). \tag{2.81}$$

For all unitary orbits $\Gamma \subset \mathcal{R}(\mathcal{H})$ one concludes: The geometric formulation of quantum mechanics makes available a 'dequantization' relation

$$f : \mathcal{B}(\mathcal{H}) \to \mathcal{F}_K(\Gamma) \tag{2.82}$$

between a $C^*$-algebra of operators to $C^*$-algebras of Kähler functions on the unitary orbits of pure quantum states.

# 3 Information Inequalities

In the following section we will consider two classes of inequalities arising in a natural way from contravariant structures of the geometric formulation of quantum mechanics. Essentially, we shall distinguish here between *entropy inequalities* in section 3.1 and *geometric inequalities* in section 3.2. The latter will fundamentally highlight a link to the *quantum Fisher information* in terms of the *quantum Cramér-Rao inequality.* Both type of inequalities may therefore be subsumed in this section under what we could call general *information inequalities.*

## 3.1 Quantum Entropy Inequalities

In the last section 2.2.3 of the previous chapter we found a dequantization relation

$$f : \mathcal{B}(\mathcal{H}) \to \mathcal{F}_K(\Gamma) \tag{3.1}$$

between operators and Kähler functions as a fundamental consequence of the geometric formulation of quantum mechanics. This has been achieved by the functions defined in (2.78) according to

$$f_A(\psi) \equiv \frac{\langle\psi|\, A\, |\psi\rangle}{\langle\psi\, |\psi\rangle}$$

together with the non-commutative star product (2.80)

$$f_A \cdot f_B|_\psi + K_{\mathcal{H}_0}(df_A, df_B)|_\psi := f_A \star f_B|_\psi$$

defined by the projectable contravariant tensor field $K_{\mathcal{H}_0}$. At this point we may consider following problem.

**Open Problem 3.1.** *Given a quantum entropy, or any other quantum (in-*

*formation) statistical measure*

$$S : D(\mathcal{H}) \to \mathbb{R}_+ \tag{3.2}$$

*on the convex subset of mixed quantum states* $D(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$*, does there exists a functional*

$$\tilde{S} : \mathcal{F}_K(\Gamma) \to \mathbb{R}_+ \tag{3.3}$$

*on* $\mathcal{F}_K(\Gamma)$ *such that the following Diagram*

$$\begin{array}{ccc} D(\mathcal{H}) & \xrightarrow{S=\tilde{S}\circ f|_{D(\mathcal{H})}} & \mathbb{R}_+ \\ {\scriptstyle \iota}\downarrow & & \uparrow{\scriptstyle \tilde{S}} \\ \mathcal{B}(\mathcal{H}) & \xrightarrow{f} & \mathcal{F}_K(\Gamma). \end{array} \tag{3.4}$$

*commutes?*

For a constructive approach to this problem, we'll need as first step a short digression on *operator convex functions*.

### 3.1.1 Operator convex functions

Operator convex functions (see e.g. [16] and references therein) are instances of general *operator functions* associating to any function

$$h : \mathbb{R} \to \mathbb{R}, \tag{3.5}$$

a map, denoted with an abuse of notation with the same letter,

$$h : u^*(\mathcal{H}) \to u^*(\mathcal{H}) \tag{3.6}$$

on Hermitian operators $A \in u^*(\mathcal{H})$, such that

$$h(A) = h(U \operatorname{Diag}\{\lambda_j\} U^\dagger) = U h(\operatorname{Diag}\{h(\lambda_j)\}) U^\dagger. \tag{3.7}$$

Operator *convex* functions are then defined with the additional property

$$h\Big(\sum_j p_j A_j\Big) \leq \sum_j p_j h(A_j), \quad p_j \in [0,1], \;\; \sum_j p_j = 1, \tag{3.8}$$

for all $A_j \in u^*(\mathcal{H})$. Note that all convex operator functions are also convex functions. Given the spectral decomposition

$$A = U \sum_j \lambda_j \left|e_j\right\rangle \left\langle e_j\right| U^\dagger \tag{3.9}$$

associated to the eigenvector basis $\{|e_j\rangle\}_{j\in J}$ one finds therefore for all $|\psi\rangle \in \mathcal{H}$

$$\begin{aligned}
\langle\psi|\, h(A)\, |\psi\rangle = & \;\; \langle\psi'| \textstyle\sum_j h(\lambda_j)\, |e_j\rangle \langle e_j\, |\psi'\rangle & (3.10)\\
= & \;\; \textstyle\sum_j |\, \langle\psi'\, |e_j\rangle\, |^2 h(\lambda_j) & (3.11)\\
\geq & \;\; h(\textstyle\sum_j |\, \langle\psi'\, |e_j\rangle\, |^2 \lambda_j) & (3.12)
\end{aligned}$$

where we set $|\psi'\rangle := U\,|\psi\rangle$. For any given resolution of the identity

$$\sum_i |\psi_i\rangle \langle\psi_i| = \mathrm{Id}_{\mathcal{H}}, \tag{3.13}$$

this implies the *Peierl's inequality* [16]

$$\mathrm{Tr}(h(A)) \geq \sum_i h(\langle\psi_i|\, A\, |\psi_i\rangle). \tag{3.14}$$

This inequality reduces to an equality if the resolution of the identity is realized by the eigenvector basis $\{|e_j\rangle\}_{j\in J}$ of $A$. The Peierl's inequality (3.14) may now be used for translating quantum entropy measures into the framework of geometric quantum mechanics as we will see in the next section.

### 3.1.2 Entropy in Geometric QM

At this point we may observe that most entropy measures arise as traces

$$S_h(\rho) := \mathrm{Tr}(h(\rho)) \tag{3.15}$$

on operator convex functions $h$. By virtue of Peierl's inequality (3.14) one finds

$$S_h(\rho) = \int_\Gamma \langle\psi_g|\, h(\rho)\, |\psi_g\rangle\, d\Gamma \geq \int_\Gamma h(\langle\psi_g|\, \rho\, |\psi_g\rangle) d\Gamma \tag{3.16}$$

for a resolution of the identity[8]

$$\int_\Gamma |\psi_g\rangle \langle\psi_g|\, d\Gamma = \mathrm{Id}_{\mathcal{H}}, \tag{3.17}$$

defined by a manifold $\Gamma \cong \mathcal{G}/\mathcal{G}_0$ of coherent states

$$|\psi_g\rangle := U(g)\, |\psi_0\rangle\,, \quad U : \mathcal{G} \to \mathcal{U}(\mathcal{H}) \tag{3.18}$$

associated to a unitary representation $U(g) \in \mathcal{U}(\mathcal{H})$. With this we arrive to an inequality

$$S_h(\rho) \geq \int_\Gamma h(f_\rho(\psi_g)) d\Gamma \tag{3.19}$$

linking quantum entropies of the form (3.20) to the geometric formulation of quantum mechanics. The pull-back of Kähler functions $f_\rho(\psi_g)$ on manifolds of coherent states coincides here with the so-called the *Husimi functions* $Q_\rho(\alpha) \equiv f_\rho(\psi_g)$ with $\alpha$ denoting a parametrization of the coherent states manifold $\Gamma$. Actually, Kähler functions are more general then Husimi functions as they may involve Kähler functions $f_{h(\rho)}(\psi_g)$ associated with convex operator functions $h$. With this generalized setting we conclude the following.

[8] We'll hide constant factors depending on the irreducible unitary representation in the measure $d\Gamma$. Bloch-coherent states induced by irreducible representations of $SU(2)$, for instance, would imply an additional factor $(2j+1)/4\pi$ [16].

**Proposition 3.2.** *For all quantum entropy measures*

$$S_h : D(\mathcal{H}) \to \mathbb{R}_+, \quad S_h(\rho) := \mathrm{Tr}(h(\rho)) \tag{3.20}$$

*defined by a trace on a convex operator function $h : \mathbb{R}_+ \to \mathbb{R}_+$ and for all manifolds $\Gamma \subset \mathcal{R}(\mathcal{H})$ of coherent states $|\psi_g\rangle := U(g)\,|\psi_0\rangle$ there exists two functionals*

$$\tilde{S}_h(f_\rho) := \int_\Gamma h(f_\rho(\psi_g))d\Gamma, \tag{3.21}$$

$$\tilde{S}(f_{h(\rho)}) := \int_\Gamma f_{h(\rho)}(\psi_g)d\Gamma, \tag{3.22}$$

*on Kähler functions $f_\rho, f_{h(\rho)} \in \mathcal{F}_K(\Gamma)$ such that the diagram in (3.4),*

$$\begin{array}{ccc} D(\mathcal{H}) & \xrightarrow{S_h} & \mathbb{R}_+ \\ {\scriptstyle\iota}\big\downarrow & & \big\uparrow{\scriptstyle \tilde{S}\geq\tilde{S}_h} \\ \mathcal{B}(\mathcal{H}) & \xrightarrow{f} & \mathcal{F}_K(\Gamma). \end{array} \tag{3.23}$$

*commutes exactly for (3.22) and approximately for (3.21) according to the inequality*

$$S_h(\rho) \geq \tilde{S}_h(f_\rho). \tag{3.24}$$

### 3.1.3 Wehrl-Inequality

For the operator convex function $h(x) \equiv x \ln x$ we may directly apply Proposition 3.2 to the von Neumann entropy

$$S_{vN}(\rho) := -S_h(\rho) = -\mathrm{Tr}(\rho \ln \rho), \tag{3.25}$$

by taking into account a sign reversion. According to Proposition 3.2 we may identify the von Neumann entropy by the functional

$$S_{vN}(\rho) = -\tilde{S}(f_{\rho \ln \rho}) = -\int_\Gamma f_{\rho \ln \rho}(\psi_g)d\Gamma \tag{3.26}$$

The corresponding approximation functional (3.21) recovers the *Wehrl-Entropy*

$$S_W(\rho) := -\tilde{S}_h(f_\rho) = -\int_\Gamma f_\rho(\psi_g)\ln f_\rho(\psi_g)d\Gamma \tag{3.27}$$

implying by virtue of (3.24) the *Wehrl-Inequality* [63]

$$S_{vN}(\rho) \leq S_W(f_\rho) \tag{3.28}$$

**Remark 3.3.** *According to the Lieb-conjecture one finds the minimum of the Wehrl-entropy given for pure states $\rho_\psi \equiv |\psi\rangle\langle\psi|$ iff $\psi$ is a coherent state vector [64].*

Here we propose another inequality being induced by the Hermitian tensor field $K_{\mathcal{H}_0}$ within the definition of the non-commutative product of Kähler functions in (2.80). For this purpose we consider the replacement of the von Neumann entropy by the functional $-S_{vN}(\rho) = \tilde{S}(f_{\rho\ln\rho})$ and find

$$\begin{aligned}
\tilde{S}(f_{\rho\ln\rho}) &= \int_\Gamma f_{\rho\ln\rho}(\psi_g)d\Gamma & (3.29)\\
&\geq \int_\Gamma f_{\rho\ln\rho}(\psi_g) - f_\rho(\psi_g)f_{\ln\rho}(\psi_g)d\Gamma & (3.30)\\
&= \int_\Gamma K(df_\rho(\psi_g), df_{\ln\rho}(\psi_g))d\Gamma, & (3.31)
\end{aligned}$$

where we used (2.79) in the last equality. In conclusion we find

$$S_{vN}(\rho) \leq -\int_\Gamma K_{\mathcal{H}_0}(df_\rho(\psi_g), df_{\ln\rho}(\psi_g))d\Gamma. \tag{3.32}$$

This last relation outlines a possible link to the inverse problem of recovering quantum relative entropies from quantum Fisher information metrics. As a matter of fact, the latter may be seen related here for pure states to the contraction with the contravariant structure $K_{\mathcal{H}_0}$.

### 3.1.4 Rény-Wehrl-Inequalities

Approximations of different orders to the von Neumann entropy may arise by a class of quantum entropies known as *quantum Rény entropies.* They are defined by

$$R_q(\rho) := \frac{1}{1-q}\ln(\mathrm{Tr}(\rho^q)), \quad q \in \mathbb{R}_+ \tag{3.33}$$

and have the property

$$\lim_{q\to 1} R_q(\rho) = S_{vN}(\rho). \tag{3.34}$$

The operator function $h(x) = -x^q$ is operator convex for $q \in [0,1]$ [16]. Thus the corresponding translation into functionals on Kähler functions on a coherent state manifold becomes

$$\begin{aligned} R_q(\rho) = &\ \tfrac{1}{1-q}\ln\left(\int_\Gamma f_{\rho^q}(\psi_g)d\Gamma\right) && (3.35)\\ \leq &\ \tfrac{1}{1-q}\ln\left(\int_\Gamma f_\rho^q(\psi_g)d\Gamma\right) && (3.36)\\ := &\ R_q^W(f_\rho) && (3.37)\end{aligned}$$

which recovers the *Rény-Wehrl entropies* and its associated *Rény-Wehrl inequalities* (compare also [16], p. 287). A lower bound family of inequalities may be induced here by (2.79) where we find for any fixed $q$ and all partitions $q = \alpha + \beta$

$$R_q(\rho) \geq \frac{1}{1-q}\ln\left(\int_\Gamma K_{\mathcal{H}_0}(df_{\rho^\alpha}(\psi_g), df_{\rho^\beta}(\psi_g))d\Gamma\right). \tag{3.38}$$

## 3.2 Geometric Inequalities

So far we considered inequalities derived from the Peierl's inequality (3.14) which reflected a particular property of operator convex functions. This inequality gave rise to quantum entropy inequalities being very naturally described in the geometric formulation of quantum mechanics.

In the following section we shall focus on another type of inequality being itself of geometric nature. The derivation is based on a recent discussion[9] made by Facchi, Marmo and Ventriglia [65].

The basic idea takes into account a short digression on some basic inequalities derived from inner products on tensor spaces. After that digression, we shall focus on inequalities derived from from inner products on tensorial structures as being particularly available in geometric quantum mechanics. At that point we shall find as a consequence the Schrödinger-Robertson inequality and a quantum version of the *Cramér Rao inequality* as being centrally used in quantum estimation problems.

### 3.2.1 Inner products on tensor spaces

Let $V$ be a linear space over the field of real or complex numbers endowed with an inner product $\langle \cdot \,|\cdot\rangle$, inducing a positive norm

$$\|v\|^2 = \langle v \,|v\rangle \geqslant 0 \tag{3.39}$$

for all $v \in V$. Now, we may consider the tensor product space $V \otimes V$ which itself is again a linear space on which the original inner product and the associated norm may be extended according to

$$\|v_1 \otimes v_2\|^2 = \langle v_1 \otimes v_2 \,|v_1 \otimes v_2\rangle \equiv \langle v_1 \,|v_1\rangle \, \langle v_2 \,|v_2\rangle \tag{3.40}$$

[9]At this point I would like to gratefully acknowledge once again all participants.

inducing a corresponding inequality

$$\|v_1 \otimes v_2\|^2 = \langle v_1 \otimes v_2 \,|v_1 \otimes v_2\rangle \geqslant 0. \tag{3.41}$$

Such an inequality turns out to be true for *all* elements in $u \in V \otimes V$ as one can check by considering a linear expansion $u = \sum_{j,k} c^{jk} e_j \otimes e_k$ in an orthonormal tensor product basis[10].

In this setting we shall identify the linear subspace $V \wedge V$ of antisymmetric contravariant 2-tensors and find as an immediate consequence

$$\|v_1 \wedge v_2\|^2 = \langle v_1 \wedge v_2 \,|v_1 \wedge v_2\rangle \geqslant 0. \tag{3.42}$$

This last inequality is equivalent with the *Cauchy-Schwarz inequality*

$$\|v_1\|^2 \|v_2\|^2 - |\, \langle v_1 \,|v_2\rangle \,|^2 \geqslant 0 \tag{3.43}$$

as one can easily check when applying the anti-symmetric tensor product expansion $v_1 \wedge v_2 := \frac{1}{2}(v_1 \otimes v_2 - v_2 \otimes v_1)$ in (3.42). As an important tool for dealing with calculations, we shall identify here the determinant of a $2 \times 2$ matrix

$$\|v_1 \wedge v_2\|^2 = \frac{1}{2} \det \begin{pmatrix} \langle v_1 \,|v_1\rangle & \langle v_1 \,|v_2\rangle \\ \langle v_2 \,|v_1\rangle & \langle v_2 \,|v_2\rangle \end{pmatrix} \geqslant 0 \tag{3.44}$$

which admits a straight forward generalization to the determinant of a $n \times n$ matrix

$$\|v_1 \wedge v_2 .. \wedge v_n(p)\|^2 = \frac{1}{n!} \det(\{\langle v_j \,|v_k\rangle\}_{j,k \in J}) \geqslant 0 \tag{3.45}$$

when applying the inner product on $V^{\otimes n}$ on the linear subspace $V^{\wedge n}$ of contravariant antisymmetric n-tensors.

---

[10]Written explicitly, we have

$$\langle u \,|u\rangle = \sum_{j,k,i,l} \bar{c}^{jk} c^{il} \langle e_j \otimes e_k \,|e_i \otimes e_l\rangle = \sum_{j,k,i,l} \bar{c}^{jk} c^{il} \delta_{jk} \delta_{il} = \sum_{i,k} |c^{ik}|^2 \geqslant 0.$$

**Remark 3.4.** *By starting on the dual vector space $V^*$, we shall find the same inequalities on the corresponding tensor space $(V^*)^{\wedge n}$ of covariant antisymmetric n-tensors.*

### 3.2.2 Tensor field contractions

By promoting tensors to tensor *fields*, and by promoting Euclidean or Hilbert spaces to real Riemannian or complex Hilbert manifolds $M$ respectively, we may recover on each tangent space $T_pM \cong V$ the inequality relations of the previous section. This can directly be seen as follows. A covariant Riemannian[11] tensor field $\eta$ admits a contraction with a vector field

$$v : M \to TM,\ p \mapsto (p, v(p)) \tag{3.46}$$

full filling the inequality

$$\|v(p)\|^2 \equiv \langle v(p)\ |v(p)\rangle_p \equiv \eta_p(v(p), v(p)) \geqslant 0. \tag{3.47}$$

In the same lines as in the previous section we may identify extended inequalities involving higher order tensor fields

$$v_1 \otimes v_2 : p \mapsto v_1 \otimes v_2(p) \in T_pM \otimes T_pM \tag{3.48}$$

yielding

$$\begin{aligned} \|v_1 \otimes v_2(p)\|^2 &= \langle v_1 \otimes v_2(p)\ |v_1 \otimes v_2(p)\rangle_p \\ &\equiv \eta_p(v_1(p), v_1(p))\eta_p(v_2(p), v_2(p)) \geqslant 0. \end{aligned} \tag{3.49}$$

For 1-forms $\alpha : M \to T^*M,\ p \mapsto (p, \alpha(p))$, we have a corresponding contraction with a contravariant Riemannian structure

$$\|\alpha(p)\|^2 \equiv G_p(\alpha(p), \alpha(p)) \geqslant 0 \tag{3.50}$$

[11] The following discussion is also valid for Hermitian tensor fields on complex manifolds.

which may be generalized to a family of inequalites

$$\|\alpha_1 \wedge \alpha_2 .. \wedge \alpha_n(p)\|^2 = \frac{1}{n!} \det(\{G_p(\alpha_j(p), \alpha_k(p))\}_{j,k\in J}) \geqslant 0 \tag{3.51}$$

defined by n-forms $\alpha_1 \wedge \alpha_2 .. \wedge \alpha_n$. Hence, to any given family of $k$ differentiable functions on $M$ there is a non-negative valued quantity defined by

$$\|df_1 \wedge df_2 .. \wedge df_n(p)\|^2 = \frac{1}{n!} \det(\{G_p(df_j(p), df_k(p))\}_{j,k\in J}) \geqslant 0. \tag{3.52}$$

Now, let us apply this setting within the geometric formulation of quantum mechanics as follows. Recall in this regard the fundamental relation between algebraic and geometric structures captured in (2.74) by

$$f_{A\cdot B}(\psi) = (G + i\Omega)(df_A(\psi), df_B(\psi)), \tag{3.53}$$

which encodes via the Hermitian structure $G + i\Omega$ the non-commutative product of two Hermitian operators $A, B$ on a Hilbert space $\mathcal{H}$ in terms of a contraction on 1-forms generated by real valued quadratic functions

$$f_A(\psi) \equiv \mathrm{Tr}(\rho_\psi A) \equiv \rho_\psi(A), \quad \rho_\psi := \frac{|\psi\rangle \langle\psi|}{\langle\psi\,|\psi\rangle}, \tag{3.54}$$

defined on $\mathcal{H}_0$. Given a finite set of Hermitian operators $\{A_j\}_{j\in J}$ on $\mathcal{H}$, we shall therefore find a direct generalization of (3.52) according to

$$\|df_{A_1} \wedge df_{A_2} .. \wedge df_{A_n}(\psi)\|^2 = \frac{1}{n!} \det(\{(G + i\Omega)(df_{A_j}(\psi), df_{A_k}(\psi))\}_{j,k\in J}) \geqslant 0. \tag{3.55}$$

At this point we consider (3.53) to conclude

$$\|df_{A_1} \wedge df_{A_2} .. \wedge df_{A_n}\|^2 = \frac{1}{n!} \det(\{(f_{A_j\cdot A_k}(\psi))\}_{j,k\in J}) \geqslant 0, \tag{3.56}$$

that is

$$\frac{1}{n!}\det(\{\rho_\psi(A_j \cdot A_k)\}_{j,k\in J}) \geqslant 0. \tag{3.57}$$

This inequality stays valid under the affine transformation map

$$A \mapsto \widetilde{A} = A - \rho_\psi(A)\mathrm{Id}_{\mathcal{H}} \tag{3.58}$$

and becomes rewritten in terms covariances of two operators

$$\rho_\psi(\widetilde{A_j} \cdot \widetilde{A_k}) = \mathrm{Cov}_{\rho_\psi}(A_j, A_k). \tag{3.59}$$

The latter relation follows directly from

$$\rho_\psi(\widetilde{A_j}\widetilde{A_k}) = \rho_\psi\Bigg((A_j - \rho_\psi(A_j)\mathrm{Id}_{\mathcal{H}})(A_k - \rho_\psi(A_k)\mathrm{Id}_{\mathcal{H}})\Bigg)$$

$$= \rho_\psi\Bigg(A_jA_k + \rho_\psi(A_j)\rho(A_k)\mathrm{Id}_{\mathcal{H}} - \rho_\psi(A_j)A_k - \rho_\psi(A_k)A_j\Bigg)$$

$$= \rho_\psi(A_jA_k) - \rho_\psi(A_j)\rho_\psi(A_k), \tag{3.60}$$

where we used the normalization $\rho_\psi(\mathrm{Id}_{\mathcal{H}}) = 1$. This coincides with the contraction $K_{\mathcal{H}_0}(df_{A_j}, df_{A_k})|_\psi$ on the contravariant tensor field $K_{\mathcal{H}_0}$ defined in (2.79) as being projectable on the space of complex rays $\mathcal{R}(\mathcal{H})$ when considering 1-forms generated by $f_{A_j}$.

In conclusion, the determinant of a $n \times n$ covariance matrix associated to a pure quantum state $\rho_\psi$ applied on a finite set of Hermitian operators $\{A_j\}_{j\in J}$ full fills

$$\frac{1}{n!}\det(\{\mathrm{Cov}_{\rho_\psi}(A_j, A_k)\}_{j,k\in J}) \geqslant 0. \tag{3.61}$$

This is obviously true for any subset $\{A_j\}_{j\in J'} \subset \{A_j\}_{j\in J}$. In particular all principal minors of a $n \times n$ covariance matrix are therefore positive, making available Sylvester's criterion to conclude

$$\{\mathrm{Cov}_{\rho_\psi}(A_j, A_k)\}_{j,k\in J} \geqslant 0. \tag{3.62}$$

Within the framework of geometric quantum mechanics this is equivalent with

$$\{K_{\mathcal{H}_0}(df_{A_j}(\psi), df_{A_k}(\psi))\}_{j,k\in J} \geqslant 0. \tag{3.63}$$

As we will show in the next section, this inequality implies the *Robertson-Schrödinger inequality* when applied to the pullback tensor induced by a *Weyl systems*.

### 3.2.3 The Robertson-Schrödinger inequality

The Fubini Study metric related pullback structure induced by a Weyl systems has been identified in section 2.1.6 according to

$$Cov_{\rho_\psi}((R(X_j)R(X_k))dv^j \otimes dv^k \tag{3.64}$$

By the virtue of (3.62) we shall find here the inequality

$$Cov_{\rho_\psi}((R(X_j)R(X_k)) \equiv \sigma(\rho_\psi) + \frac{i}{4a^2}\omega \geqslant 0, \tag{3.65}$$

which is equivalent that all principal minors of the covariance matrix are positive. A 2n-form

$$df_{\tilde{R}(X_1)} \wedge df_{\tilde{R}(X_2)} .. \wedge df_{\tilde{R}(X_{2n})}(\rho_\psi) \quad \text{with} \;\; f_{\tilde{R}(X_j)}(\rho_\psi) := \mathrm{Tr}(\rho_\psi \tilde{R}(X_j)) \tag{3.66}$$

implies in particular according to section 3.2.2

$$\|df_{\tilde{R}(X_1)} \wedge df_{\tilde{R}(X_2)} .. \wedge df_{\tilde{R}(X_{2n})}(\rho_\psi)\|^2 = \frac{1}{(2n)!}\det(\{\rho_\psi(\tilde{R}(X_j)\tilde{R}(X_k)\}_{j,k\in J}) \geqslant 0, \tag{3.67}$$

that is

$$\frac{1}{(2n)!}\det\left(\sigma(\rho_\psi) + \frac{i}{4a^2}\omega\right) \geqslant 0. \tag{3.68}$$

In the simplest case of one degree of freedom modeled on $\mathcal{H} \cong L^2(\mathbb{R})$, the set of the Weyl-map generating elements are given up to an imaginary unit by two Hermitian operators

$$R(X_1) := Q, \quad R(X_2) := P \tag{3.69}$$

related to position and momentum respectively. The Weyl-map induced pull-back tensor coefficient matrix defines in dependence of the fiducial state vector $\psi \in L^2(\mathbb{R})$ the covariance matrix

$$\sigma(\rho_\psi) + i\Omega = \begin{pmatrix} (\triangle_{\rho_\psi} Q)^2 & \mathrm{Cov}^s_{\rho_\psi}(Q,P) + \frac{i}{4a^2} \\ \mathrm{Cov}^s_{\rho_\psi}(P,Q) - \frac{i}{4a^2} & (\triangle_{\rho_\psi} P)^2 \end{pmatrix} \tag{3.70}$$

with the variance $(\triangle_{\rho_\psi} A)^2 := \mathrm{Cov}_{\rho_\psi}(A,A) = \mathrm{Var}_{\rho_\psi}(A)$ and the symmetrized covariances $\mathrm{Cov}^s_{\rho_\psi}(A,B) = \mathrm{Cov}^s_{\rho_\psi}(B,A)$. By taking into account the 2-form $df_{\tilde{Q}} \wedge df_{\tilde{P}}(\rho_\psi)$, we find according to (3.67)

$$\|df_{\tilde{Q}} \wedge df_{\tilde{P}}(\rho_\psi)\|^2 = \frac{1}{2} \det\left(\sigma(\rho_\psi) + \frac{i}{4a^2}\omega\right) \geqslant 0, \tag{3.71}$$

yielding the *Robertson-Schrödinger inequality*

$$(\triangle_{\rho_\psi} Q)^2 (\triangle_{\rho_\psi} P)^2 \geqslant \frac{1}{16a^4} + \mathrm{Cov}^s_{\rho_\psi}(Q,P)^2. \tag{3.72}$$

Due to $\mathrm{Cov}^s_{\rho_\psi}(Q,P)^2 \geqslant 0$ it implies the *Heisenberg inequality*

$$\triangle_{\rho_\psi} Q \, \triangle_{\rho_\psi} P \geqslant \frac{1}{4a^2} = \begin{cases} \frac{1}{2} \text{ canonical convention} \\ \frac{1}{4} \text{ q-optical convention.} \end{cases} \tag{3.73}$$

In contrast to the Robertson-Schrödinger inequality, we shall emphasize that the Heisenberg inequality is *not* invariant under symplectic transformations [66]. The invariance of the Robertson-Schrödinger inequality under symplectic transformations follows directly from the invariance of the determinant of the general inequality (3.67),

$$\sigma(\rho_\psi) + \frac{i}{4a^2}\omega \geqslant 0,$$

under volume preserving transformations including symplectic transformations as special case.

To make the difference between the Heisenberg inequality and the Robertson-Schrödingier inequality explicit we compute according to (2.54) the symplectic eigenvalues of

$$i\omega \cdot \sigma = i \begin{pmatrix} 0 & -\frac{1}{4a^2} \\ \frac{1}{4a^2} & 0 \end{pmatrix} \cdot \begin{pmatrix} \sigma_{1,1} & \sigma_{1,2} \\ \sigma_{1,2} & \sigma_{2,2} \end{pmatrix} = \begin{pmatrix} -\frac{\sigma_{1,2}}{4a^2} & -\frac{\sigma_{2,2}}{4a^2} \\ \frac{\sigma_{1,1}}{4a^2} & \frac{\sigma_{1,2}}{4a^2} \end{pmatrix} \tag{3.74}$$

and find

$$\mathrm{spec}(i\omega \cdot \sigma) = \left\{ \pm\frac{1}{4}\sqrt{\frac{\sigma_{1,2}^2 - \sigma_{1,1}\sigma_{2,2}}{a^4}} \right\}. \tag{3.75}$$

Hence, the Heisenberg-inequality fails to provide a symplectic invariant due to the lack of the symmetric covariances $\sigma_{1,2}^2$.

### 3.2.4 Quantum Cramér Rao Inequality

Let $\mathcal{M}$ be a subset in a Hilbert space $\mathcal{H}$ provided by any smooth curve of state vectors

$$[0,1]_{\subset\mathbb{R}} \to \mathcal{H}, \quad \lambda \mapsto |\psi_\lambda\rangle\,. \tag{3.76}$$

On this subset of Hilbert space vectors

$$\{\psi_\lambda\}_{\lambda\in[0,1]} = \mathcal{M} \subset \mathcal{H} \tag{3.77}$$

we may assume an operator $\frac{\partial}{\partial\lambda}$ to be well defined. Moreover we may consider a further operator $\hat{\lambda}$ on $\mathcal{M}$, such that

$$f_{\hat{\lambda}}(\psi_\lambda) = \frac{\langle\psi_\lambda|\,\hat{\lambda}\,|\psi_\lambda\rangle}{\langle\psi_\lambda\,|\psi_\lambda\rangle} \equiv \lambda \tag{3.78}$$

holds. By applying on this setting the geometric inequalities as derived in section 3.2.2 according to (3.62) - (3.63) one finds

$$K_{\mathcal{H}_0}(df_{\frac{\partial}{\partial\lambda}}, df_{\frac{\partial}{\partial\lambda}})|_{\psi_\lambda} K_{\mathcal{H}_0}(df_{\hat{\lambda}}, df_{\hat{\lambda}})|_{\psi_\lambda} - |K_{\mathcal{H}_0}(df_{\frac{\partial}{\partial\lambda}}, df_{\hat{\lambda}})|_{\psi_\lambda}|^2 = \tag{3.79}$$

$$Var_{\psi_\lambda}(\frac{\partial}{\partial\lambda})Var_{\psi_\lambda}(\hat{\lambda}) - |Cov_{\psi_\lambda}(\frac{\partial}{\partial\lambda}, \hat{\lambda})|^2 \geq \ 0 \tag{3.80}$$

which turns out to imply a quantum version of the *Cramér Rao inequality* [67]

$$Var_{\psi_\lambda}(\lambda) \geq \frac{|Cov_{\psi_\lambda}(\frac{\partial}{\partial\lambda}, \lambda)|^2}{Var_{\psi_\lambda}(\frac{\partial}{\partial\lambda})} \geq \frac{1}{Var_{\psi_\lambda}(\frac{\partial}{\partial\lambda})} \tag{3.81}$$

as being used in quantum estimation theory [68] by associating $\hat{\lambda}$ to an *unbiased estimator*[12] of the curve parameter $\lambda$. The implication of the Cramér Rao inequality in the above quantum version may be subsumed as follows. No matter which estimator one uses to approach a parameter in a quantum mechanical experimental setting, the variance of any estimator will always be bounded by the variance

$$Var_{\psi_\lambda}\left(\frac{\partial}{\partial\lambda}\right) = K_{\mathcal{H}_0}(df_{\frac{\partial}{\partial\lambda}}, df_{\frac{\partial}{\partial\lambda}})|_{\psi_\lambda} \tag{3.82}$$

$$= f_{\frac{\partial}{\partial\lambda}\frac{\partial}{\partial\lambda}}(\psi_\lambda) + f_{\frac{\partial}{\partial\lambda}} f_{\frac{\partial}{\partial\lambda}}(\psi_\lambda) \tag{3.83}$$

$$= \langle\partial_\lambda\psi_\lambda\,|\partial_\lambda\psi_\lambda\rangle - \langle\psi_\lambda\,|\partial_\lambda\psi_\lambda\rangle^2 . \tag{3.84}$$

Actually, this turns out to be the pullback coefficient of of the Fubini Study metric related pullback structure $\kappa_{\mathcal{H}_0}$ as defined in (2.11). As matter of fact,

[12]The basic definition is assured here by (3.78). We will focus on this notion once again more in detail in section 5.

one finds

$$\kappa_{\mathcal{M}} \equiv \langle \partial_\lambda \psi_\lambda \left| \partial_\lambda \psi_\lambda \right\rangle d\lambda \otimes d\lambda - \langle \psi_\lambda \left| \partial_\lambda \psi_\lambda \right\rangle^2 d\lambda \otimes d\lambda. \tag{3.85}$$

As being related to the pullback to the Fubini Study metric, it 'closes a circle' to the *quantum Fisher information* as described in section 1.4. Actually, the quantum Fisher information defined on pure states reduces to the classical Fisher information for Lagriangian submanifolds as we have seen in seen in section 2.1.3 (see also [7]). In conclusion, any quantum Cramér Rao inequality for estimating the 'position' on a curve of pure quantum states reduces to the classical Cramér Rao inequality. Note that the above quantum Cramér Rao inequality, even though being 'classical' may be seen as a generalization of the Heisenberg inequality as it allows to considers the pullback on both unitarily and non-unitarily generated curves.

# 4 Entanglement Monotones

Symmetries of Hamiltonian dynamical systems are related to constants of motion by virtue of Noether's theorem [54]. Constants of motion provide in turn a method to reduce the numbers of degrees of freedom. For instance, invariance under the symmetry group of time translations $(\mathbb{R}, +)$ implies the identification of energy as constant of motion, and therefore the identification of time independent energy surfaces embedded in a phase space $\mathbb{R}^{2n}$, via the inverse $H^{-1}(E) \subset \mathbb{R}^{2n}$ of the underlying Hamiltonian function. A Hamiltonian system generated by $H$ will thus be constrained for all times on exactly one of these surfaces in dependence of the initial state $v_0 \in \mathbb{R}^{2n}$.

In the following section we shall propose a similar approach for the identification of what one could call *entanglement invariants* by virtue of functions being constant on 'entanglement surfaces' defining submanifolds of quantum states with equivalent amount of entanglement. In contrast to the foliation into constant energy surfaces defined via the inverse of a time invariant Hamilton function however, we shall tackle here an inverse problem:

*Given* a foliation of a Hilbert space into submanifolds of quantum states with same entanglement, how do we identify a function varying from leave to leave but being constant within each leave?

First of all we note that a general quantum evolution[13] will *not* keep the entanglement in a composite system $\mathcal{H}_A \otimes \mathcal{H}_B$ unchanged. The invariance of entanglement is given only for non-interacting closed subsystems indeed.

From a kinematical perspective, this implies the identification of the *local unitary subgroup*

$$SU(\mathcal{H}_A) \times SU(\mathcal{H}_B) \subset SU(\mathcal{H}_A \otimes \mathcal{H}_B) \tag{4.1}$$

as the fundamental symmetry group of entanglement. The identification of

[13] According to the geometric formulation of quantum mechanics, a unitary quantum evolution is an integral curve of an Hamiltonian vector field being a Killing vector field.

'entanglement surfaces' is therefore provided by a stratification into orbits generated by this subgroup [69, 70]. Such a stratification induces within the geometric formulation of quantum mechanics a pullback of the Fubini Study metric related structure (2.11) from $\mathcal{H} := \mathcal{H}_A \otimes \mathcal{H}_B$ to each single orbit. In the simplest case of two qubits one may identify here local unitary orbits $\Gamma_{\psi_\lambda} \subset \mathbb{C}^2 \otimes \mathbb{C}^2$ in dependence of a 1-parameter family of fiducial state vectors

$$[0,1]_{\subset\mathbb{R}} \to \mathbb{C}^2 \otimes \mathbb{C}^2, \quad \lambda \mapsto |\psi_\lambda\rangle \tag{4.2}$$

intersecting each submanifold $\Gamma_{\psi_\lambda}$ exactly once. The pullback on each single orbit $\Gamma_{\psi_\lambda}$ may therefore considered complementary to the pullback on the curve (4.2) provided by a structure as defined in the previous section in (3.85). The following subsection 4.1 will give a short review how to identify pullback structures associated to unitarily generated orbits in dependence of a given general initial state vector of a composite bipartite Hilbert space in finite but arbitrary high dimensions as discussed in [1–3]. These structures will imply of a *qualitative* characterization of entanglement making available the identification of separable and 'maximal entangled' state vectors in purely geometrical terms. Thereafter, in the following subsections 4.2 - 4.5, we shall consider recent research results [4–6] based on the possibility to identify invariant functions under the local unitary group arising as inner products on invariant tensor fields. This will lead us to a *quantitative* characterization of entanglement in tensorial terms.

## 4.1 Separability and Lagrangian entanglement

By considering the reducible representation

$$\begin{aligned} \mathcal{G} \equiv SU(n) \times SU(n) \to & SU(n^2) \\ g \equiv (g_A, g_B) \mapsto & U(g) \equiv g_A \otimes g_B = (g_A \otimes \mathbb{1}_{\mathcal{H}_B})(\mathbb{1}_{\mathcal{H}_A} \otimes g_B) \end{aligned} \tag{4.3}$$

infinitesimal generated by traceless orthonormal Hermitian matrices $\{\sigma_j\}_{j\in J}$ tensored by the identity $\mathbb{1}_{\mathcal{H}_A}$ of a subsystem

$$\begin{aligned}\mathrm{Lie}(\mathcal{G}) \equiv su(n) \oplus su(n) \to & su(n^2) \\ X_j \mapsto & iR(X_j) \equiv \begin{cases} i\sigma_j \otimes \mathbb{1} & \text{for } 1 \leq j \leq n^2 \\ \mathbb{1} \otimes i\sigma_{j-n^2} & \text{for } n^2+1 \leq j \leq 2n^2, \end{cases}\end{aligned} \tag{4.4}$$

one finds according to theorem 2.1 a pull-back tensor field on the Lie group

$$\iota_{\mathcal{G}}^* \kappa_{\mathcal{H}_0} = Cov_{\rho_\psi}(R(X_j)R(X_k))\theta^j \otimes \theta^k$$

$$= \underbrace{(\rho_\psi([R(X_j)R(X_k)]_+) - \rho_\psi(R(X_j))\rho_\psi(R(X_k)))\theta^j \odot \theta^k}_{=\iota_{\mathcal{G}}^*\eta} + i\underbrace{\rho_\psi([R(X_j)R(X_k)]_-)\theta^j \wedge \theta^k}_{=\iota_{\mathcal{G}}^*\omega}$$

which decomposes for all $\rho_\psi \in D^1(\mathbb{C}^n \otimes \mathbb{C}^n)$ into a symmetric and an anti-symmetric coefficient matrix [14]

$$(\kappa_{jk}^{\rho_\psi}) = \begin{pmatrix} (\eta_{(jk)}^{\rho_A}) & (C_{jk}^{\rho_\psi}) \\ (C_{jk}^{\rho_\psi}) & (\eta_{(jk)}^{\rho_B}) \end{pmatrix} + i \begin{pmatrix} (\omega_{[jk]}^{\rho_A}) & 0 \\ 0 & (\omega_{[jk]}^{\rho_B}) \end{pmatrix}, \tag{4.8}$$

$$\eta_{(jk)}^{\rho_A} = \frac{2}{n}\delta_{jk} + \rho_A(\sigma_r)d_{jkr} - \rho_A(\sigma_j)\rho_A(\sigma_k) \tag{4.9}$$

$$\omega_{[jk]}^{\rho_A} = \rho_A(\sigma_r)c_{jkr} \tag{4.10}$$

$$C_{jk}^{\rho_\psi} = \rho_\psi(\sigma_j \otimes \sigma_k) - \rho_\psi(\sigma_j)\rho_\psi(\sigma_k) \tag{4.11}$$

---

[14] Written explicitly one finds

$$\eta_{(jk)}^{\rho_A} = \rho_\psi([\sigma_j, \sigma_k]_+ \otimes \mathbb{1}) - \rho_\psi(\sigma_j \otimes \mathbb{1})\rho_\psi(\sigma_k \otimes \mathbb{1}) \tag{4.5}$$

$$= \rho_A([\sigma_j, \sigma_k]_+) - \rho_A(\sigma_j)\rho_A(\sigma_k) = \frac{2}{n}\delta_{jk} + \rho_A(\sigma_r)d_{jkr} - \rho_A(\sigma_j)\rho_A(\sigma_k) \tag{4.6}$$

$$\omega_{[jk]}^{\rho_A} = \rho_\psi([\sigma_j, \sigma_k]_- \otimes \mathbb{1}) = \rho_A([\sigma_j, \sigma_k]_-) = \rho_A(\sigma_r)c_{jkr} \tag{4.7}$$

by vitue of the symmetric and anti-symmetric structure constants $d_{jkr}$ and $c_{jkr}$ of the Lie algebra of $SU(n)$.

### 4.1.1 Segre embeddings seen from the Hilbert space

While the anti-symmetric part $\omega^{\rho_\psi}$ admits a splitting for all entangled fiducial state vectors, we will find such a splitting in the symmetric part $\eta^{\rho_\psi}$ according to the following necessary and sufficient condition [1,3].

**Theorem 4.1.** $\rho_\psi \in D^1(\mathbb{C}^n \otimes \mathbb{C}^n)$ *is separable iff*

$$\eta^{\rho_\psi}_{SU(n)\times SU(n)} = \eta^{\rho_A \otimes \rho_B}_{SU(n)\times SU(n)} = \eta^{\rho_A}_{SU(n)} \oplus \eta^{\rho_B}_{SU(n)} \tag{4.12}$$

Hence, a *Segre embedding* [42, 43, 71]

$$\mathcal{R}(\mathcal{H}_A) \times \mathcal{R}(\mathcal{H}_B) \hookrightarrow \mathcal{R}(\mathcal{H}_A \otimes \mathcal{H}_B) \tag{4.13}$$

becomes detectable from the point of view of the Hilbert space iff

$$C^{\rho_\psi}_{jk} = 0. \tag{4.14}$$

For general state vectors we shall remark the following geometric interpretation of the symmetric part $\eta_{SU(n)\times SU(n)}$. It encounters as pullback induced by the projection

$$SU(n) \times SU(n) \to SU(n) \times SU(n)/\mathcal{G}_{0,\psi} \tag{4.15}$$

on the orbits associated with the isotropy group

$$\mathcal{G}_{0,\psi} := \{g \in SU(n) \times SU(n) | U(g)\rho_\psi U(g)^\dagger = \rho_\psi\} \tag{4.16}$$

the complete information of a family of *Riemannian* tensor fields on each given orbit.

### 4.1.2 Symplectic orbits seen from the Hilbert space

The anti-symmetric part

$$\omega^{\rho_\psi}_{SU(n)\times SU(n)} = \rho_\psi([R(X_j), R(X_k)]_-)\theta^j \wedge \theta^k \tag{4.17}$$

splits — as indicated above and in crucial contrast to the symmetric part — for all entangled state vectors into two families of tensor fields

$$\omega^{\rho_\psi}_{SU(n)\times SU(n)} = \omega^{\rho_A}_{SU(n)} \oplus \omega^{\rho_B}_{SU(n)}. \tag{4.18}$$

Each family is defined on a corresponding $SU(n)$-subgroup of $SU(n)\times SU(n)$ by the *reduced density state dependent* anti-symmetric structures

$$\omega^{\rho_A}_{SU(n)} = \rho_\psi([\sigma_j, \sigma_k]_- \otimes \sigma_0)\theta^j \wedge \theta^k \tag{4.19}$$

$$= \rho_A([\sigma_j, \sigma_k]_-)\theta^j \wedge \theta^k, \tag{4.20}$$

and $\omega^{\rho_B}_{SU(n)} = \rho_B([\sigma_j, \sigma_k]_-)\theta^j \wedge \theta^k$ respectively.

Each of the latter anti-symmetric tensor fields may therefore identified with the quotient space projection-induced pullback of a symplectic structure $\omega^{A}_{SU(n)/\mathcal{G}_0}$ which lives on a co-adjoint unitary orbit of (reduced) density states

$$g_A \rho_A g_A^\dagger, \tag{4.21}$$

with $g_A \in SU(n)$ [72]. *Lagrangian* orbits generated by the local unitary group admit therefore a distinguished role, in particular, due to the following definition and associated implication [3, 4]:

**Definition 4.2.** $\rho_\psi \in D^1(\mathbb{C}^n \otimes \mathbb{C}^n)$ *is called Lagragian entangled if*

$$\omega^{\rho_\psi}_{SU(n)\times SU(n)} = 0.$$

**Theorem 4.3.** $\rho_\psi \in D^1(\mathbb{C}^n \otimes \mathbb{C}^n)$ *is Lagragian entangled iff the reduced* *density state is maximal mixed.*

This theorem links to the standard definition of 'maximally entangled' pure states as used in the literature [73]. Actually, the identification of maximal entangled states is *not* a priori obvious *before* having defined a measure of entanglement. As a matter of fact, the identification of the von Neumann entropy (1.8) as unique measure is established *after* rather then before the definition of maximal entanglement [19]. Such an approach may appear artificial from a general point of view indeed. In particular, why should the existence of an infinite amount of entanglement a priori be excluded? The notion of *Lagrangian entanglement*, as proposed here, evades this conceptual problem by being defined a priori without any quantitative association.

## 4.2 Inner products on tensor fields and intermediate entanglement

In the previous subsection we considered a qualitative description of entanglement by the identification of submanifolds containing either separable or Lagrangian entangled quantum state vectors. Actually, most state vectors in the composite Hilbert space turn out to be in neither of both of these two submanifolds. This can be illustrated in the simplest case of a two qubit Hilbert space

$$\mathcal{H} \equiv \mathcal{H}_A \otimes \mathcal{H}_B \cong \mathbb{C}^2 \otimes \mathbb{C}^2 \tag{4.22}$$

as follows. Any coordinate representation of a state vector in $S(\mathcal{H})$ may be transformed here by means of a local unitary transformation to a Schmidt basis decomposition

$$|\psi_\lambda\rangle = \sqrt{\lambda}\,|00\rangle + \sqrt{1-\lambda}\,|11\rangle\,, \quad \lambda \in [0,1]. \tag{4.23}$$

In this way one encounters a topological distinction into three type of local unitary orbits

$$SU(2) \times SU(2)/\mathcal{G}_{0,\lambda} \tag{4.24}$$

with the isotropy group

$$\mathcal{G}_{0,\lambda} := \{g \in SU(2) \times SU(2) | U(g)\rho_{\psi_\lambda} U(g)^\dagger = \rho_{\psi_\lambda}\} \tag{4.25}$$

in dependence of fiducial states $\rho_{\psi_\lambda} = |\psi_\lambda\rangle \langle\psi_\lambda|$ being either Lagrangian entangled ($\lambda = 1/2$), separable ($\lambda \in \{0, 1\}$) or neither Lagrangian nor separable ($\lambda \in (0, 1/2) \cup (1/2, 1)$). They yield orbits of dimensions three, four and five respectively [69, 70]. The 'dense set' of state vectors is therefore made of an foliation into five dimensional orbits of what we could call *intermediate* entangled state vectors.

In the following we'll show how to distinguish the topologically equivalent orbits of intermediate states in geometrical terms by means of the corresponding Fubini Study metric related pullback procedure we discussed so far.

**Remark 4.4.** *In higher dimensional bipartite systems there will be more then three topological inequivalent type of local unitary orbits [69, 70]. Indeed, different topological invariants like the dimension of a manifold will nevertheless be detected from the metrical pullback structures on the individual orbit. Such topological invariants may be detected in particular within our approach by the degeneracy of the pullback on the Lie group of local unitary transformation.*

For the purpose to distinguish state vectors living in different but topological equivalent orbits we may consider invariant functions under local unitary transformations provided by a (super) Hermitian inner product

$$f(\psi) := \left\langle \kappa^{\rho_\psi}_{SU(n)\times SU(n)} \middle| \kappa^{\rho_\psi}_{SU(n)\times SU(n)} \right\rangle \tag{4.26}$$

on invariant tensor fields on $SU(n) \times SU(n)$. Invariant tensor fields on general

Lie groups may be defined in a constructive way either by *invariant operator valued tensor fields* (see section 4.5) or by the pullback of the Fubini Study metric seen from the Hilbert space as considered so far. The inner product on the latter class of invariant tensor fields will be defined in general terms as follows [4].

We define for any given Lie group $\mathcal{G}$ associated with a unitary representation induced pullback structure

$$\kappa_{\mathcal{G}}^{\rho_\psi} = \kappa_{jk}^{\rho_\psi} \theta^j \otimes \theta^k$$

the inner product

$$\left\langle \kappa_{\mathcal{G}}^{\rho_\psi} \left| \kappa_{\mathcal{G}}^{\rho_\psi} \right.\right\rangle := \bar{\kappa}_{jk}^{\rho_\psi} \kappa_{rl}^{\rho_\psi} \left\langle \theta^j \otimes \theta^k \left| \theta^r \otimes \theta^l \right.\right\rangle .$$

To keep the formulas as readable as possible, we shall omit in the following the dependency on the fiducial state $\rho_\psi$. With

$$\left\langle \theta^j \otimes \theta^k \left| \theta^r \otimes \theta^l \right.\right\rangle = \left\langle \theta^j \left| \theta^r \right.\right\rangle \left\langle \theta^k \left| \theta^l \right.\right\rangle = \delta^{jr} \delta^{kl} \tag{4.27}$$

one finds then

$$\left\langle \kappa_{\mathcal{G}} \left| \kappa_{\mathcal{G}} \right.\right\rangle = \bar{\kappa}_{jk} \kappa_{rl} \delta^{jr} \delta^{kl} = \bar{\kappa}_{jk} \kappa_{jk}. \tag{4.28}$$

### 4.2.1 Inner products on higher order tensor fields

In the following we may consider a class of $\mathcal{G}$-invariant functions arising from higher order tensor fields

$$\kappa_{\mathcal{G}}^{\otimes n} := \bigotimes_{k=1}^{n} \kappa_{\mathcal{G}} \tag{4.29}$$

by virtue of their corresponding inner product

$$\left\langle \kappa_{\mathcal{G}}^{\otimes n} \left| \kappa_{\mathcal{G}}^{\otimes n} \right.\right\rangle \tag{4.30}$$

$$= \quad \left\langle \kappa_{\mathcal{G}} \left| \kappa_{\mathcal{G}} \right.\right\rangle^n . \tag{4.31}$$

This can be be illustrated in the simplest case of $n = 2$ as follows. Consider a general covariant tensor of order four

$$T := \sum_{j_1 j_2 j_3 j_4} T_{j_1 j_2 j_3 j_4} \theta^{j_1} \otimes \theta^{j_2} \otimes \theta^{j_3} \otimes \theta^{j_4}. \tag{4.32}$$

The inner product $\langle T \,|T\rangle$ on this tensor reads

$$\begin{aligned}
&\left\langle \sum_{j_1 j_2 j_3 j_4} T_{j_1 j_2 j_3 j_4} \theta^{j_1} \otimes \theta^{j_2} \otimes \theta^{j_3} \otimes \theta^{j_4} \,\middle|\, \sum_{k_1 k_2 k_3 k_4} T_{k_1 k_2 k_3 k_4} \theta^{k_1} \otimes \theta^{k_2} \otimes \theta^{k_3} \otimes \theta^{k_4} \right\rangle \\
&= \sum_{j_1 j_2 j_3 j_4} \sum_{k_1 k_2 k_3 k_4} \bar{T}_{j_1 j_2 j_3 j_4} T_{k_1 k_2 k_3 k_4} \left\langle \theta^{j_1} \otimes \theta^{j_2} \otimes \theta^{j_3} \otimes \theta^{j_4} \,\middle|\, \theta^{k_1} \otimes \theta^{k_2} \otimes \theta^{k_3} \otimes \theta^{k_4} \right\rangle \\
&= \sum_{j_1 j_2 j_3 j_4} \sum_{k_1 k_2 k_3 k_4} \bar{T}_{j_1 j_2 j_3 j_4} T_{k_1 k_2 k_3 k_4} \delta^{j_1 k_1} \delta^{j_2 k_2} \delta^{j_3 k_3} \delta^{j_4 k_4} \\
&= \sum_{j_1 j_2 j_3 j_4} \bar{T}_{j_1 j_2 j_3 j_4} T_{j_1 j_2 j_3 j_4}.
\end{aligned} \tag{4.33}$$

A special tensor of order four may arise from the tensor product of two tensors of order two

$$\begin{aligned}
T &\equiv \left( \sum_{j_1 j_2} \kappa_{j_1 j_2} \theta^{j_1} \otimes \theta^{j_2} \right) \otimes \left( \sum_{j_3 j_4} \kappa_{j_3 j_4} \theta^{k_3} \otimes \theta^{j_4} \right) \\
&= \sum_{j_1 j_2} \sum_{j_3 j_4} \kappa_{j_1 j_2} \kappa_{j_3 j_4} \theta^{j_1} \otimes \theta^{j_2} \otimes \theta^{j_3} \otimes \theta^{j_4}.
\end{aligned} \tag{4.34}$$

In this case, the tensor coeffiecients in (4.38) factorize into tensor coefficients of order two according to

$$T_{j_1 j_2 j_3 j_4} \equiv \kappa_{j_1 j_2} \kappa_{j_3 j_4}. \tag{4.35}$$

Note that this is not true for general tensors of order four.

Thus, by applying the inner product (4.33) on the *special case* given by the tensor in (4.34), we find

$$\langle T \,|T\rangle = \sum_{j_1j_2j_3j_4} \bar{T}_{j_1j_2j_3j_4} T_{j_1j_2j_3j_4} = \sum_{j_1j_2j_3j_4} \bar{\kappa}_{j_1j_2}\bar{\kappa}_{j_3j_4}\kappa_{j_1j_2}\kappa_{j_3j_4}$$

$$= \sum_{j_1j_2j_3j_4} \bar{\kappa}_{j_1j_2}\kappa_{j_1j_2}\bar{\kappa}_{j_3j_4}\kappa_{j_3j_4} = \sum_{j_1j_2} \bar{\kappa}_{j_1j_2}\kappa_{j_1j_2} \sum_{j_3j_4} \bar{\kappa}_{j_3j_4}\kappa_{j_3j_4} \tag{4.36}$$

$$= \langle \kappa_{\mathcal{G}} \,|\kappa_{\mathcal{G}}\rangle \langle \kappa_{\mathcal{G}} \,|\kappa_{\mathcal{G}}\rangle = \langle \kappa_{\mathcal{G}} \,|\kappa_{\mathcal{G}}\rangle^2 . \tag{4.37}$$

The generalization is straight forward and goes as follows. The inner product $\langle T \,|T\rangle$ on a general covariant tensor[15]

$$T := T_{j_1j_2...j_m}\theta^{j_1} \otimes \theta^{j_2} ... \otimes \theta^{j_m} \tag{4.38}$$

of order $m$ reads

$$\bar{T}_{j_1..j_m} T_{k_1..k_m} \left\langle \theta^{j_1} \otimes .. \otimes \theta^{j_m} \,\middle|\theta^{k_1} \otimes .. \otimes \theta^{k_m} \right\rangle$$

$$= \bar{T}_{j_1..j_m} T_{k_1..k_m} \delta_{j_1k_1} ... \delta_{j_mk_m} = \bar{T}_{j_1..j_m} T_{j_1..j_m}. \tag{4.39}$$

Now, we consider a tensor of even order $m = 2n$ constructed from the n-th tensor product of order two tensors

$$T \equiv (T_{j_1j_2}\theta^{j_1} \otimes \theta^{j_2})^{\otimes n}$$

$$= T_{j_1j_2} T_{j_3j_4} .. T_{j_{m-1}j_m} \theta^{j_1} \otimes \theta^{j_2} \otimes \theta^{j_3} \otimes \theta^{j_4} ... \theta^{j_{m-1}} \otimes \theta^{j_m}$$

$$= \prod_{r=1}^{n} T_{j_{2r-1}j_{2r}} \bigotimes_{r=1}^{n} \theta^{j_{2r-1}} \otimes \theta^{j_{2r}}. \tag{4.40}$$

The tensor coefficients in (4.38) factorize in this special case (in each term

[15]If not differently stated, we shall from now on use the Einstein convention by summing over same indices.

of the sum over same indices) into tensor coefficients of order two

$$T_{j_1 j_2 \dots j_m} = \prod_{r=1}^{n} T_{j_{2r-1} j_{2r}}. \tag{4.41}$$

Hence, with the inner product (4.39) one concludes

$$\langle T \mid T \rangle = \bar{T}_{j_1 .. j_m} T_{j_1 .. j_m} = \prod_{r=1}^{n} \bar{T}_{j_{2r-1} j_{2r}} \prod_{r=1}^{n} T_{j_{2r-1} j_{2r}}$$

$$= \prod_{r=1}^{n} \bar{T}_{j_{2r-1} j_{2r}} T_{j_{2r-1} j_{2r}} = \prod_{r=1}^{n} \langle T \mid T \rangle = \langle T \mid T \rangle^{n}. \tag{4.42}$$

This proofs the inner product relation (4.31). As a consequence, we may apply the inner product on the tensor product of the symmetric part and the tensor product of the anti-symmetric part separately and find

$$\left\langle \eta_{\mathcal{G}}^{\otimes n} \,\middle|\, \eta_{\mathcal{G}}^{\otimes n} \right\rangle = \langle \eta_{\mathcal{G}} \mid \eta_{\mathcal{G}} \rangle^{n}, \tag{4.43}$$

$$\left\langle \omega_{\mathcal{G}}^{\otimes n} \,\middle|\, \omega_{\mathcal{G}}^{\otimes n} \right\rangle = \langle \omega_{\mathcal{G}} \mid \omega_{\mathcal{G}} \rangle^{n}. \tag{4.44}$$

**Remark 4.5.** *Actually, this will be not the case whenever one considers the inner product on the symmetrization of the tensor products on the symmetric part, and correspondently, the inner product on the anti-symmetrization of the tensor products on the anti-symmetric part respectively. Such an approach may be seen related to the notion of Poincaré invariants which are constructed from higher order anti-symmetric structures* $\omega^{\wedge n} := \omega \wedge \omega ... \wedge \omega$ *[74].*

## 4.3 Entanglement monotones on two qubits

In the following section we shall apply the inner product on higher order tensor fields to the case of the local unitary group

$$\mathcal{G} \equiv SU(n) \times SU(n) \tag{4.45}$$

with particular focus on $n \equiv 2$. In this case we may map a family of fiducial states

$$\rho_\lambda = |\psi_\lambda\rangle \langle\psi_\lambda| \tag{4.46}$$

associated to a family of state vectors in a Schmidt basis decomposition

$$|\psi_\lambda\rangle = \sqrt{\lambda}\,|00\rangle + \sqrt{1-\lambda}\,|11\rangle\,, \quad \lambda \in [0,1] \tag{4.47}$$

to a corresponding family of induced pullback tensor fields

$$\kappa^{\rho_\lambda}_{SU(2)\times SU(2)} = \eta^{\rho_\lambda}_{SU(2)\times SU(2)} + i\omega^{\rho_\lambda}_{SU(2)\times SU(2)} \tag{4.48}$$

on the Lie group $SU(2) \times SU(2)$. The inner products (4.43) and (4.44) applied here to the higher order tensor products of the symmetric and the anti-symmetric part of (4.48) respectively, establish the main result of the present chapter: We find *entanglement monotones* for the inner product on the tensor products of the symmetric part and *purity monotones* for the inner product on the tensor products of the anti-symmetric part as illustrated within an appropriate normalization in figure 1 for the first five tensor power orders. As normalization we use in this regard the factors

$$\frac{1}{(2\mathrm{Dim}(SU(2)\times SU(2))^n} = \frac{1}{12^n}, \tag{4.49}$$

for the inner products (4.43) and the factors

$$\frac{1}{\mathrm{Dim}(S^2\times S^2)^n} = \frac{1}{4^n}. \tag{4.50}$$

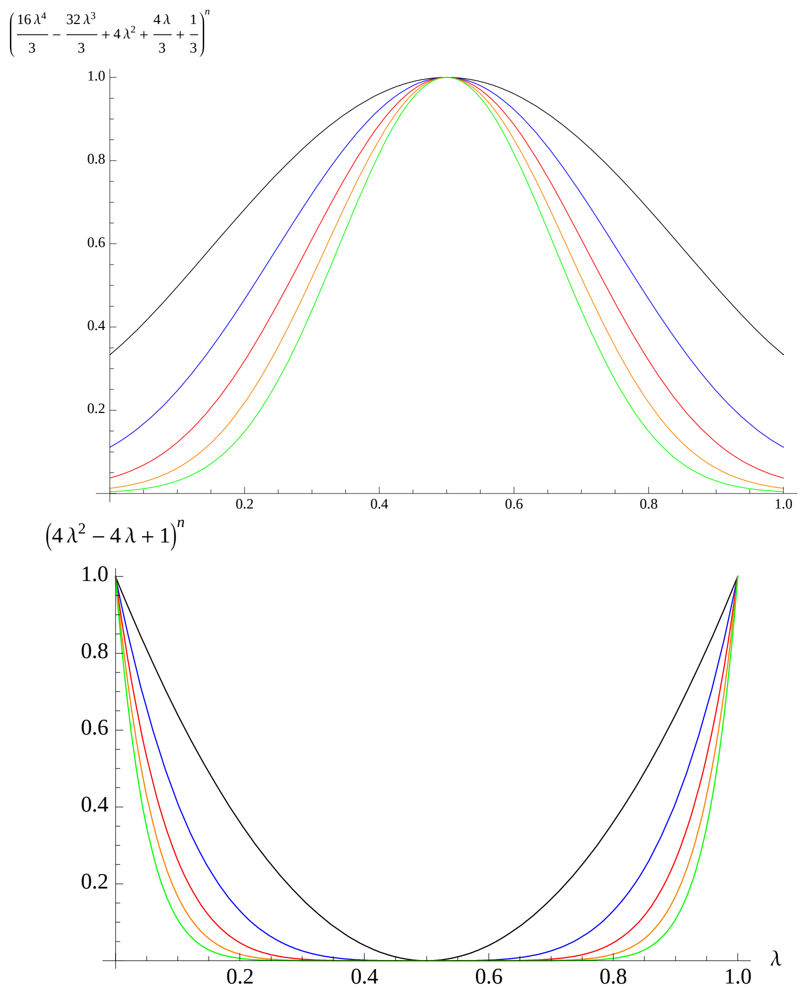



Figure 1: A class of entanglement and purity monotones constructed from $SU(2) \times SU(2)$ invariant tensor fields $\eta^{\otimes n}_{SU(2)\times SU(2)}$ and $\omega^{\otimes n}_{SU(2)\times SU(2)}$. The monotones arise here by considering an inner product yielding the invariant functions (4.51) and (4.52) respectively. The black curve corresponds in each case to $n = 1$ and the following colored curves corresponds to higher order tensor field inner products with $n \in \{2, 3, 4, 5\}$. The entanglement monotones admit in contrast to the purity monotones a normalization in dependence of the tensor order $n$.

for the inner products (4.44). The resulting functions are higher order polynomials on the Schmidt coefficient $\lambda$

$$\begin{aligned}\frac{1}{12^n}\left\langle \eta^{\rho_\lambda}_{SU(2)\times SU(2)} \middle| \eta^{\rho_\lambda}_{SU(2)\times SU(2)} \right\rangle^n &= \left(\tfrac{16\lambda^4}{3} - \tfrac{32\lambda^3}{3} + 4\lambda^2 + \tfrac{4\lambda}{3} + \tfrac{1}{3}\right)^n & (4.51)\\ \frac{1}{4^n}\left\langle \omega^{\rho_\lambda}_{SU(2)\times SU(2)} \middle| \omega^{\rho_\lambda}_{SU(2)\times SU(2)} \right\rangle^n &= \left(4\lambda^2 - 4\lambda + 1\right)^n . & (4.52)\end{aligned}$$

establishing a *quantitative* justification for the identification between *maximal entangled states* and *Lagrangian entangled* states ($\lambda = 1/2$) on the value '1' for the entanglement monotones and the value '0' for the purity monotones respectively. The purity monotone is normalized for separable states ($\lambda \in \{0, 1\}$) to '1'. In contrast, we find in the entanglement monotone a normalization

$$\frac{1}{12^n}\left\langle \eta^{\rho_\lambda}_{SU(2)\times SU(2)} \middle| \eta^{\rho_\lambda}_{SU(2)\times SU(2)} \right\rangle^n |_{\lambda\in\{0,1\}} = \frac{1}{3^n}, \tag{4.53}$$

being *dependent on the tensor field order* $n$. We may therefore recover for $n \to \infty$ the 'standard normalization' for separable states. This indicates that the inner products on $n$-order tensor products of the symmetric tensor field provide an *approximation* to a 'bona fide' entanglement measure with increasing $n$. This approximation will provide an advantage in contrast to standard entanglement measures when testing their quantum estimation efficiency as shown later on in section 5.

## 4.4 Towards a generalized algorithm

Before we proceed, let us outline a possible bigger picture into which we may set the considered approach so far by closing a *circle* to the introduction of the present chapter 4. For this purpose let us recall the Hamiltonian

$$H(q, p) = \frac{1}{2m}p^2 + kq^2. \tag{4.54}$$

of a classical Harmonic oscillator. Here one finds associated energy ellipses

$$H^{-1}(E) := \Gamma_E \subset \mathbb{R}^2 \tag{4.55}$$

in dependence of the mass $m$ and the constant $k$. If we set $m = k = 1$ we may identify the ellipses with circles $S^2_{\sqrt{E}}$ each with a radius proportional to the energy $H(q,p) = E \in \mathbb{R}_+$. The set of energy circles provides a foliation

$$\mathbb{R}^2_0 \cong \bigcup_{E \in \mathbb{R}_+} \Gamma_E = \bigcup_{E \in \mathbb{R}_+} S^1_{\sqrt{E}} \tag{4.56}$$

of the phase space into circles with different radius $\sqrt{E}$. The radius may be related in this regard to a parametrization of the set of orbits given by the quotient

$$\mathbb{R}^2_0/SO(2) \cong \mathbb{R}_+ \times S^1/SO(2) \cong \mathbb{R}_+ \tag{4.57}$$

suggesting following commutative diagram.

$$\begin{tikzcd} \mathbb{R}^2_0 \arrow[r, "\cong"] \arrow[d, "SO(2)"'] & \bigcup \Gamma_E \arrow[d, "H(q,p)=p^2+q^2"] \\ \mathbb{R}^2_0/SO(2) \arrow[r, "\cong"] & \mathbb{R}_+. \end{tikzcd}$$

This diagram appears very similar to what we found here in the case of the space of state vectors in a composite Hilbert space

$$S(\mathcal{H}) = S(\mathbb{C}^2 \otimes \mathbb{C}^2) = \bigcup_{\lambda} \Gamma_\lambda \tag{4.58}$$

being stratified into local unitary orbits of *entanglement surfaces* $\Gamma_\lambda$. The corresponding *entanglement function* relating to each orbit a constant, but distinguished value has been provided here by an inner product

$$f(\psi_\lambda) := \left\langle T^{\otimes k}_{SU(2)\times SU(2)} \middle| T^{\otimes kn}_{SU(2)\times SU(2)} \right\rangle \tag{4.59}$$

on higher order tensor products of invariant symmetric and anti-symmetric pullback tensor fields

$$T_{SU(2)\times SU(2)} := \begin{cases} \eta_{SU(2)\times SU(2)} \\ \omega_{SU(2)\times SU(2)}. \end{cases} \tag{4.60}$$

Any entanglement monotone on the set of Schmidt coefficients

$$\begin{aligned} \epsilon : \triangle_1 \rightarrow & \quad [0,1] \\ \lambda \mapsto & \quad \epsilon(\lambda) \end{aligned} \tag{4.61}$$

may therefore be (re-) constructed from such a function if

$$f\left(\Gamma_{\psi_\lambda}\right) =_! \epsilon(\lambda), \tag{4.62}$$

that is, if the following diagram commutes

$$\begin{array}{ccc} S(\mathcal{H}) & \xrightarrow{\cong} & \bigcup \Gamma_\lambda \\ {\scriptstyle SU(2)^{\times 2}}\Big\downarrow & & \Big\downarrow {\scriptstyle f(\psi)=\left\langle T^{\otimes k}_{SU(2)\times SU(2)} \middle| T^{\otimes k}_{SU(2)\times SU(2)} \right\rangle} \\ \triangle_1 & \xrightarrow{\epsilon} & \mathbb{R}_+. \end{array}$$

In the previous section we have shown that there exits a family of entanglement monotones where this diagram commutes. Interestingly, this family has not been discussed in the literature so far and provides therefore a new class of entanglement monotones.

At this point we outline possible generalizations to higher dimensions and mixed states. Indeed, it may become clear that the inner product of any tensor product on the symmetric part

$$\left\langle \eta^{\otimes k}_{SU(n)\times SU(n)}(\rho_\psi) \middle| \eta^{\otimes k}_{SU(n)\times SU(n)}(\rho_\psi) \right\rangle \tag{4.63}$$

should in principle provide an entanglement monotone candidate *for all* finite dimensions $n \in \mathbb{N}$ as being invariant under $SU(n) \times SU(n)$. This invariance will be discussed from a constructive point of view in more detail in the next section.

A generalization to convex combinations of pure states to the regime of mixed states finally, may be given by corresponding extensions to convex combination pure state entanglement monotones (see e.g. [42, 43] and references therein)

$$\mathcal{E}(\rho) := \sum_{\inf\ \rho=\sum p_j \rho_j} p_j \epsilon(\rho_j) \tag{4.64}$$

$$\equiv_! \sum_{\inf\ \rho=\sum p_j \rho_j} p_j \left\langle \eta^{\otimes k}_{SU(n)\times SU(n)}(\rho_j) \middle| \eta^{\otimes k}_{SU(n)\times SU(n)}(\rho_j) \right\rangle . \tag{4.65}$$

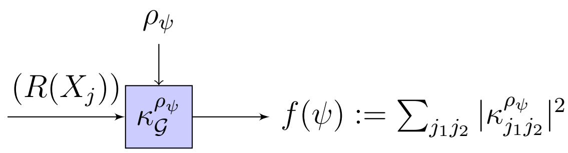
Actually, it appears not a trivial task to perform the computation here due to the infimum in the sum at the first glance. However, an exception where it is known how to compute the generalization from a pure to a mixed state entanglement measure is given by the *concurrence* [75, 76]. A digression on a link to the concurrence measure in geometrical terms will be discussed in the following section.

## 4.5 Invariant operator valued tensor fields (IOVTs) and mixed states entanglement

So far we outlined an entanglement characterization *algorithm* based on invariant tensor fields on the Lie group $\mathcal{G} = U(n) \times U(n)$, which 'replaces' functions

$$\begin{aligned} \epsilon : \triangle_{n-1} \to &\quad [0,1] \\ \lambda \mapsto &\quad \epsilon(\lambda) \end{aligned} \tag{4.66}$$

on Schmidt-coefficients by functions on pullback tensor-coefficients:




Similar to the case of pure states, we shall also identify within the generalized regime of mixed states entanglement monotone candidates by functions

$$f : D(\mathcal{H}_A \otimes \mathcal{H}_B) \rightarrow \mathbb{R}_+ \tag{4.67}$$

which are invariant under the local unitary group of transformations $U(\mathcal{H}_A) \times U(\mathcal{H}_B)$ [77]. In this necessary strength, we propose in the following entanglement monotones candidates by taking into account constant functions on local unitary orbits of entangled quantum states, arising from *invariant operator valued tensor fields* (IOVTs) on $U(\mathcal{H}_A) \times U(\mathcal{H}_B)$ as considered recently on general matrix Lie groups $\mathcal{G}$ [4]. Let us review the basic construction.

### 4.5.1 The basic construction

Given a unitary representation

$$U : \mathcal{G} \rightarrow U(\mathcal{H}), \tag{4.68}$$

we may identify an anti-Hermitian operator-valued left-invariant 1-form

$$-U(g)^{-1} dU(g) \equiv iR(X_j)\theta^j \tag{4.69}$$

on $\mathcal{G}$, where the operator $iR(X_j)$ is associated with the representation of the Lie algebra Lie($\mathcal{G}$). In this way, we may construct higher order invariant operator valued tensor fields

$$-U(g)^{-1} dU(g) \otimes U(g)^{-1} dU(g) = R(X_j)R(X_k)\theta^j \otimes \theta^j, \tag{4.70}$$

on $\mathcal{G}$ by taking into account the representation as being equivalently defined by means of the representation of the enveloping algebra of the Lie algebra in the operator algebra $\mathcal{A}$ :=End($\mathcal{H}$). More specific, any element $X_j \otimes X_k$ in the enveloping algebra becomes associated with a product

$$R(X_j)R(X_k) \in \mathcal{A} := \mathrm{End}(\mathcal{H}), \tag{4.71}$$

where $\mathcal{A}$, may denote the vector space of a $C^*$-algebra. At this point, we may evaluate each one of these products by means of dual elements

$$\rho \in \mathcal{A}^*, \tag{4.72}$$

according to

$$\rho(R(X_j)R(X_k)) \equiv \mathrm{Tr}(\rho\, R(X_j)R(X_k)) \in \mathbb{C}, \tag{4.73}$$

yielding a quantum state dependent tensor field

$$\rho(R(X_j)R(X_k))\theta^j \otimes \theta^k \tag{4.74}$$

on the group manifold. By taking the k-th product of invariant operator-valued left-invariant 1-forms

$$-U(g)^{-1}dU(g) \otimes U(g)^{-1}dU(g) \otimes ... \otimes U(g)^{-1}dU(g), \tag{4.75}$$

we shall find a representation R-dependent IVOT of order $k$

$$\theta_R := \left( \prod_{a=1}^{k} R(X_{i_a}) \right) \bigotimes_{a=1}^{k} \theta^{i_a} \tag{4.76}$$

on a Lie group $\mathcal{G} = U(n) \times U(n)$. After evaluating it with a *mixed* quantum state

$$\theta_R \mapsto \rho(\theta_R) := \theta_R^\rho = \rho\Big(\prod_{a=1}^{k} R(X_{i_a})\Big) \bigotimes_{a=1}^{k} \theta^{i_a}$$

one may identify $\mathcal{G}$-invariant functions via an inner product on tensor fields

$$f^R(\rho) := \langle \theta_R^\rho \,|\theta_R^\rho \rangle \tag{4.77}$$

as constructed in section 4.2.

### 4.5.2 Purity, concurrence and covariance measures

In particular, for $k = n = 2$ [4], we recover in this way the purity and the concurrence related measures involving a spin-flip transformed state $\tilde{\rho}$ by considering inner product combinations of symmetric and anti-symmetric tensor fields

$$\eta_R^\rho := \rho([R(X_j), R(X_k)]_+)\theta^j \odot \theta^k \tag{4.78}$$

$$\omega_R^\rho := \rho([R(X_j), R(X_k)]_-)\theta^j \wedge \theta^k, \tag{4.79}$$

according to

$$\frac{1}{8}\big( \langle \eta_R^\rho \,|\eta_R^\rho \rangle + (-1)^s \langle \omega_R^\rho \,|\omega_R^\rho \rangle \big) - \frac{1}{2} = \begin{cases} \mathrm{Tr}(\rho^2) & \text{for } s = 0 \\ \mathrm{Tr}(\rho\tilde{\rho}) & \text{for } s = 1. \end{cases} \tag{4.80}$$

In more general terms, one may introduce *R-classes* of entanglement monotone candidates by taking into account polynomials

$$f_k^R(\rho) := \sum_n a_n \langle \theta_R^\rho \,|\theta_R^\rho \rangle^n , \quad \theta_R^\rho := \rho\Big(\prod_{a=1}^{k} R(X_{i_a})\Big) \bigotimes_{a=1}^{k} \theta^{i_a}.$$

The case

$$\tilde{R}(X_j) = R(X_j) - \rho(R(X_j))\mathbb{1}, \tag{4.81}$$

recovers for IOVTs of order $k = 2$, a class of separability criteria associated with covariance matrices (CMs) (Gittsovich et al. 2008) by means of a *CM-tensor field*

$$\theta^{\rho}_{\tilde{R}} = (\rho(R(X_j)R(X_k)) - \rho(R(X_j))\rho(R(X_k))\theta^j \otimes \theta^k. \tag{4.82}$$

An open problem in the field of CM-ctiteria is provided by the question how to find an extension to quantitative statements [78]. A possible approach could be provided here by taking into account a $\tilde{R}$-*class* of entanglement monotone-candidates by considering

$$f_2^{\tilde{R}}(\rho) = \sum_n a_n \left\langle \theta^{\rho}_{\tilde{R}} \middle| \theta^{\rho}_{\tilde{R}} \right\rangle^n .$$

To give an example, we consider the function

$$f_2^{\tilde{R}}(\rho) \equiv \left\langle \theta^{\rho}_{\tilde{R}} \middle| \theta^{\rho}_{\tilde{R}} \right\rangle \tag{4.83}$$

applied to a family of 2-parameter states on a composite Hilbert space of two qubits given by

$$\rho_{x,\alpha_0} := x \left|\alpha_0\right\rangle \left\langle\alpha_0\right| + (1-x)\rho^*, \quad \left|\alpha_0\right\rangle := \cos(\alpha_0)\left|11\right\rangle + \sin(\alpha_0)\left|00\right\rangle \tag{4.84}$$

and find a possible approximation to the concurrence measure

$$\max[\lambda_4 - \lambda_3 - \lambda_2 - \lambda_1, 0], \quad \lambda_j \in \mathrm{Spec}(\rho\widetilde{\rho}). \tag{4.85}$$

Both functions are plotted in figure 2.

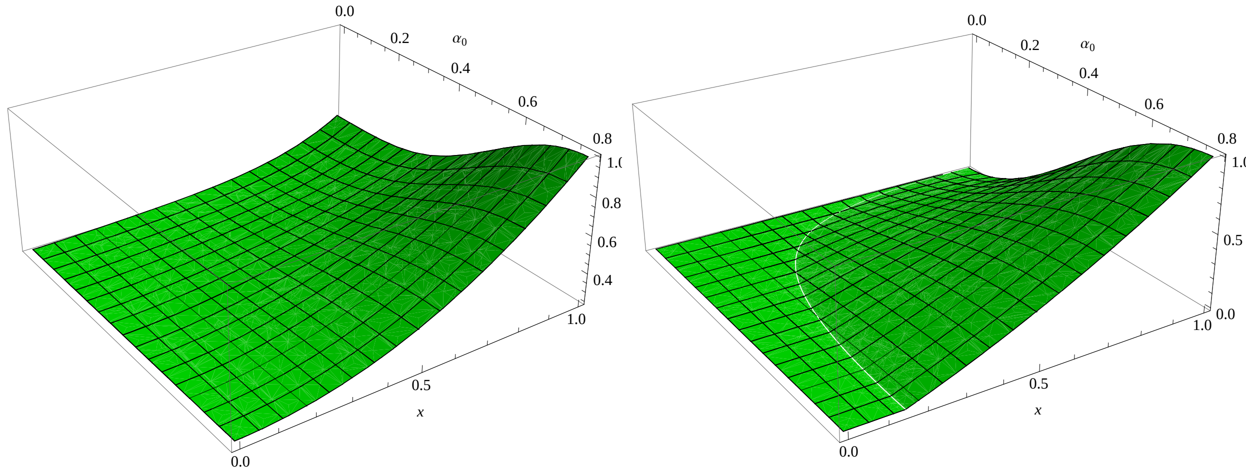



Figure 2: The function $f_2^{\tilde{R}}(\rho)$ gives rise to a possible approximation (left) to the concurrence measure (right) applied to a family of 2-parameter states on a composite Hilbert space of two qubits.

# 5 Entanglement Estimation

In the previous section we formulated an algorithm for a tensorial characterization of entanglement both from a qualitative and quantitative point of view. This algorithm had been modeled by a pairing between a quantum state $\rho$ and a finite set of observables $\{R(X_j)\}_{j\in J}$ giving rise to a function on 'classical' tensor coefficients

$$\begin{tikzcd} & \rho \arrow[d] & \\ {} \arrow[r, "(R(X_j))"] & \boxed{\kappa^{\rho}_{\mathcal{G}}} \arrow[r] & f(\rho) := \sum_{j_1 j_2} |\kappa^{\rho}_{j_1 j_2}|^2 \end{tikzcd}$$

on a given Lie group $\mathcal{G}$ associated with a unitary representation. Interestingly, the tensor coefficients

$$\kappa^{\rho}_{j_1 j_2} = Cov_{\rho}(R(X_{j_1})R(X_{j_2})) \tag{5.1}$$

provide covariances and therefore empirically accessible quantities. With other words, these type of entanglement monotones, as suggested in particular by the geometric formulation of quantum mechanics here, may in principle be reconstructed empirically from an *incomplete* set of observables, without the need of a complete tomographic reconstruction of the quantum state.

In contrast, designing experimental settings able to achieve a *quantitative* description of entanglement is a current challenge, even with the most modern quantum experimental devices available today [8]. One fundamental reason may be related to the fact that standard entanglement monotones are in general non-linear functions on the convex body of quantum states [47,77], rather than ordinary expectation values of 'quantum observables' associated with Hermitian operators in a Hilbert space. As indicated above, a possible approach to this problem could be provided, on the one hand, by a tomographic reconstruction of the quantum state involving a series of measurements associated with a complete set of observables (see, e.g., [79] for a review on quantum tomography). On the other hand, while a quantum state is supposed to contain the complete information about a physical system, a sufficient amount of information about entanglement may be extracted by means of an *incomplete* set of observables as in the case of the here considered pullback tensor coefficients using a reducible Lie algebra representation associated to $SU(n) \times SU(n)$ rather than the irreducible representation of $SU(n^2)$.

Parallel to our approach we shall mention at this point similar remarks in the literature on standard entanglement measures like on the *concurrence* (4.85) [75, 76], which may be experimentally recovered from four — rather than fifteen — parameters associated with the Bloch representation of a bipartite qubit density matrix [80]. Moreover, an optimal experimental setting for quantifying pure state entanglement – based on the purity – which requires the reconstruction of only three parameters associated with local quantum observables, has been proposed in [81].

Actually, there exists a general framework on the optimization bounds of quantum experiments based on *quantum state estimation theory* [47, 82, 83] which originated in the seminal work of Helstrom in the 1960s [68]. In contrast, the specialization of this general framework to what could be called *quantum entanglement estimation theory* is a relatively young field of research (see [8] and references therein). Recent works have focused on the optimization bounds in terms of quantum information measures, like the Kullback mutual information [84], the fidelity [85] – and finally – the quantum Fisher information [8].

Crucially, in the simplest non-trivial case of a 1-parameter family of bipartite pure qubit states in the Schmidt decomposition, several standard entanglement measure fail to provide an efficient quantum Fisher information estimations – in particular – for *weakly* entangled states [8].

In this section, we propose an efficient estimation in the regime of weak entanglement by taking into account the family of alternative entanglement monotones as constructed in the previous section 4. This will allow us to merge the idea of a covariant tensorial characterization of entanglement with the contravariant tensorial identification of geometric inequalities of section 3 into one unified geometric framework.

This section is organized as follows. In subsection 5.1 we review the basic idea of quantum state estimation by using the *Cramèr-Rao inequality* as derived in section 3.2.4. Thereafter, in sections 5.1.1 and 5.1.2, we consider in particular the Schmidt coefficient estimation and the entanglement estimation of two entangled qubits, as recently discussed in [8]. Next, in subsection 5.1.3, we focus on the corresponding estimation of the purity. In subsection 5.2 we will address the estimation of the entanglement and alternative purity measures as constructed in the previous section. We give our conclusions and an outlook in section 5.3.

## 5.1 Quantum state estimation

The most elementary quantum estimation problem may be formulated as follows.

- Given an unknown quantum state vector $|\psi_\lambda\rangle$ on a curve of state vectors parametrized by $\lambda \in [0,1]$. Can we extract the parameter $\lambda$ from $|\psi_\lambda\rangle$ at small measurement cost?

To tackle this question one defines to any *finite* number of measurements, let's say a sample $\{x_j\}$, an *estimator* of $\lambda$ by a map

$$\{x_j\} \mapsto \hat{\lambda}(\{x_j\}) \in [0,1]. \tag{5.2}$$

The efficiency of the estimator is then given by the *mean square error*

$$\mathbb{E}_\lambda((\hat{\lambda}(\{x_j\}) - \lambda)^2). \tag{5.3}$$

The error coincides for *unbiased estimators*

$$\mathbb{E}_\lambda(\hat{\lambda}) = \lambda \tag{5.4}$$

with the variance

$$\mathrm{Var}(\hat{\lambda}) = \mathbb{E}_\lambda(\hat{\lambda}^2) - \mathbb{E}_\lambda(\hat{\lambda})^2. \tag{5.5}$$

The efficiency of all unbiased estimators is then bounded by the *quantum-Cramér-Rao inequality*

$$\mathrm{Var}(\hat{\lambda}) \geq \frac{1}{\kappa_\lambda}, \tag{5.6}$$

where $\kappa_\lambda$ denotes the coefficient of the pullback of the Fubini Study metric on a given curve of Hilbert space vectors $|\psi_\lambda\rangle$ as seen from the Hilbert space. This inequality has been derived in (3.81) according to section 3.2.4 and closes therefore a circle from the geometric formulation of quantum mechanics to the concept of *quantum Fisher information* being identified here with the Fubini

Study metric related pullback coefficient $\kappa_\lambda$. The physical interpretation of (5.6) becomes directly available when taking into account the *relative error* [47] depending on

- the number $M \in \mathbb{N}$ of measurements and
- the *signal to noise ratio* $\lambda^2/\mathrm{Var}(\hat{\lambda})$

The relative error is defined by

$$\delta^2 := \frac{\mathrm{Var}(\hat{\lambda})}{M\lambda^2} \equiv \frac{1}{M}\frac{\text{Noise}}{\text{Signal}}. \tag{5.7}$$

The quantum Cramer-Rao inequality (5.6) implies thus a $\kappa_\lambda$-bounded *minimum* number of measurements

$$M_\delta \geq \frac{1}{\lambda^2\delta^2}\frac{1}{\kappa_\lambda} \tag{5.8}$$

required for achieving an estimation of $\lambda$ with a given fixed relative error $\delta^2$.

### 5.1.1 Schmidt coefficient estimation

The fundamental example we'll focus on in the following is the case where $\lambda$ coincides with a Schmidt coefficient parametrizaing a curve of entangled two qubit state vectors

$$|\psi_\lambda\rangle = \sqrt{\lambda}\,|00\rangle + \sqrt{1-\lambda}\,|11\rangle\,, \quad \lambda \in [0,1] \tag{5.9}$$

in a composite Hilbert space $\mathcal{H} \cong \mathbb{C}^2 \otimes \mathbb{C}^2$. For this purpose we compute the pullback of the Fubini-Study metric

$$\langle \partial_\lambda \psi_\lambda \,|\partial_\lambda \psi_\lambda\rangle \, d\lambda \otimes d\lambda - \langle \psi_\lambda \,|\partial_\lambda \psi_\lambda\rangle^2 \, d\lambda \otimes d\lambda \equiv \kappa_\lambda d\lambda \otimes d\lambda \tag{5.10}$$

on $\{|\psi_\lambda\rangle\}_{\lambda\in[0,1]}$ as seen from the Hilbert space. The coefficient yields in this case the quantum Fisher information

$$\kappa_\lambda = \frac{1}{\lambda - \lambda^2} \tag{5.11}$$

As one can see, the *minimum* number of measurements (5.8)

$$M_\delta(\lambda) = \frac{1}{\lambda^2\delta^2}\frac{1}{\kappa_\lambda} = \frac{\lambda - \lambda^2}{\lambda^2\delta^2} \sim \frac{1}{\lambda\delta^2} \tag{5.12}$$

diverges for $\lambda \to 0$. With other words, the estimation becomes inefficient for all states close to the separable state vector $|11\rangle$.

### 5.1.2 Linear entropy estimation

At this point we may note that any entanglement measure on $|\psi_\lambda\rangle$ relates to a measure

$$\epsilon : [0,1] \to [0,1], \quad \lambda \mapsto \epsilon(\lambda). \tag{5.13}$$

on the parameter space of Schmidt coefficients. Such a measure induces a non-linear parameter transformation on the pullback coefficient

$$\kappa_\epsilon \equiv \kappa_{\lambda(\epsilon)}(\partial_\epsilon\lambda(\epsilon))^2 \tag{5.14}$$

provided by the covariant transformation property of the pullback tensor

$$\begin{aligned}
\kappa_\lambda d\lambda \otimes d\lambda = & \quad \kappa_{\lambda(\epsilon)} d\lambda(\epsilon) \otimes d\lambda(\epsilon) & (5.15)\\
= & \quad \kappa_{\lambda(\epsilon)}\partial_\epsilon\lambda(\epsilon)d\epsilon \otimes \partial_\epsilon\lambda(\epsilon)d\epsilon & (5.16)\\
= & \quad \kappa_{\lambda(\epsilon)}(\partial_\epsilon\lambda(\epsilon))^2 d\epsilon \otimes d\epsilon & (5.17)\\
\equiv & \quad \kappa_\epsilon d\epsilon \otimes d\epsilon. & (5.18)
\end{aligned}$$

As a consequence, one finds a *distinguished* minimum number of measurements

$$M_\delta(\epsilon) = \frac{1}{\epsilon^2\delta^2}\frac{1}{\kappa_\epsilon} \tag{5.19}$$

required for achieving an estimation of $\epsilon(\lambda)$ rather then an estimation of $\lambda$. Indeed, similar to the Schmidt coefficient estimation, it implies again a divergence, in particular for vanishing values of $\epsilon$

$$\lim_{\epsilon\to 0} M_\delta(\epsilon) \to \infty \tag{5.20}$$

whenever one identifies the measure $\epsilon$ with the *linear entropy*

$$\epsilon(\lambda) := 2\big(1 - \mathrm{Tr}\big((\rho_\lambda^A)^2\big)\big) = 4\lambda(1-\lambda) \tag{5.21}$$

or the *negativity* $\epsilon_N(\lambda) := \sqrt{\epsilon(\lambda)}$. In conclusion, the estimation of standard entanglement measures related to the linear approximation of the von Neumann entropy becomes *inefficient* in the regime of *weak* entanglement [8].

### 5.1.3 Purity estimation

At this point we may ask for the estimation efficiency of the purity

$$\mathrm{Tr}\big((\rho_\lambda^A)^2\big) \tag{5.22}$$

defining in contrast to the linear entropy

$$2\big(1 - \mathrm{Tr}\big((\rho_\lambda^A)^2\big)\big)$$

an entanglement *anti-monoton* admitting non-vanishing values close to one in the regime of weak entanglement. The purity measure (5.22) for the reduced density state

$$\rho_\lambda^A = \begin{pmatrix} \lambda & 0 \\ 0 & 1-\lambda \end{pmatrix} \tag{5.23}$$

associated with the family of entangled state vectors (5.9) reads

$$\epsilon(\lambda) := \mathrm{Tr}((\rho_\lambda^A)^2) = \lambda^2 + (1-\lambda)^2 = 1 - 2\lambda + 2\lambda^2. \tag{5.24}$$

To compute the induced parameter transformation on the quantum Fisher information (5.14), we need to identify the inverse function solutions of the purity

$$\lambda(\epsilon) := \frac{1}{2}\left(1 \pm \sqrt{2\epsilon - 1}\right). \tag{5.25}$$

Both solutions yield according to (5.14) the parameter-transformed quantum Fisher information

$$\kappa_\epsilon = -\frac{1}{4\epsilon^2 - 6\epsilon + 2}. \tag{5.26}$$

However, this implies a *negative* number of measurements

$$M_{\delta\equiv 1}(\epsilon) = -4 + \frac{6}{\epsilon} - \frac{2}{\epsilon^2} \tag{5.27}$$

within the estimation (5.19), appearing beyond a physical interpretation. Hence, the purity does not solve the problem of estimating weakly entangled qubits in an efficient way.

## 5.2 Estimation of inner products on tensor fields

A class of entanglement monotones may be constructed in terms of functionals on the linearization of the von Neumann entropy

$$
\begin{aligned}
S_{vN}(\lambda) \quad &= \mathrm{Tr}(\rho_\lambda^A \ln \rho_\lambda^A) = \textstyle\sum_j \lambda_j \ln \lambda_j \\
\downarrow \quad & \\
S_{vN}^{linear}(\lambda) \quad &:= 1 - \mathrm{Tr}((\rho_\lambda^A)^2) = 1 - \textstyle\sum_j \lambda_j^2 \\
\downarrow \quad & \\
f\big(S_{vN}^{linear}(\lambda)\big) \quad &= \begin{cases} 2S_{vN}^{linear}(\lambda) \\ \sqrt{2S_{vN}^{linear}(\lambda)} \\ 1 - S_{vN}^{linear}(\lambda) \end{cases}
\end{aligned}
\tag{5.28}
$$

providing the linear entropy, the negativity and the purity as (anti-) monotones on Schmidt coefficients respectively. This class of entanglement monotones constructed in this way highlights a clear advantage compared to the von Neumann entropy as it does not involve the the diagonalization of the reduced density state (or equivalently, the singular value decomposition into Schmidt coefficients of the state vector associated to the product Hilbert space). However, as we have seen in the previous section, such monotones may not become accessible in experiments as soon we enter into the regime of weak entanglement.

At this point we observe the following. The linear entropy is directly related to the pullback coefficient $\kappa_\lambda$ of the Fubini metric by

$$
S_{vN}^{linear}(\lambda) = \frac{2}{\kappa_\lambda} \tag{5.29}
$$

due to (5.11) in

$$
2(1 - \mathrm{Tr}((\rho_\lambda^A)^2) = 4\lambda(1-\lambda) =_{(5.11)} \frac{4}{\kappa_\lambda}. \tag{5.30}
$$

The quantum Cramér Rao inequality (5.6) may thus be rewritten for a family of two entangled qubits as entropy inequality

$$Var(\lambda) \geq \frac{1}{2} S_{vN}^{linear}(\lambda) \tag{5.31}$$

and may therefore be linked to the class of quantum entropy inequalities as considered in section 3.1. Actually, the relation (5.29) between linear entropy and the pullback coefficient of the Fubini Study metric highlights a link to the beginning of section 1.4 considering any quantum Fisher information metric as the Hessian of some quantum relative entropy.

Our 'inverse trip' from geometric quantum mechanics to quantum information may therefore come to an end at this point. Actually, one of the most important implications of geometric quantum mechanics for quantum information theory has still to be discussed. Indeed, $\kappa_\lambda$ is the pullback coefficient of the Fubini Study metric on a particular chosen 1-dimensional entangled state submanifold. In the following we ask: What happens if we estimate the entanglement monotones provided by the inner products

$$\epsilon_n := \quad \tfrac{1}{12^n} \left\langle \eta_{SU(2)\times SU(2)}^{\otimes n} \middle| \eta_{SU(2)\times SU(2)}^{\otimes n} \right\rangle \tag{5.32}$$

$$\mu_n := \quad \tfrac{(-1)^n}{4^n} \left\langle \omega_{SU(2)\times SU(2)}^{\otimes n} \middle| \omega_{SU(2)\times SU(2)}^{\otimes n} \right\rangle \tag{5.33}$$

on the tensor products of the symmetric and anti-symmetric part of the pullback tensor fields

$$\kappa_{SU(2)\times SU(2)} = \eta_{SU(2)\times SU(2)} + i\omega_{SU(2)\times SU(2)} \tag{5.34}$$

related to the orbits of the symmetry group of entanglement $SU(2) \times SU(2)$ constructed in section 4.2?

For this purpose we apply the procedure of section 5.1.2 by first identifying the inverse functions $\lambda(\epsilon_n)$ and $\lambda(\mu_n)$ of the monotones (4.51) and

(4.52),

$$\frac{1}{12^n}\mathrm{Tr}(\eta^2)^n = \left(\frac{16\lambda^4}{3} - \frac{32\lambda^3}{3} + 4\lambda^2 + \frac{4\lambda}{3} + \frac{1}{3}\right)^n \tag{5.35}$$

$$\frac{(-1)^n}{4^n}\mathrm{Tr}(\omega^2)^n = \left(4\lambda^2 - 4\lambda + 1\right)^n \tag{5.36}$$

as solutions of the equations

$$\left(\frac{16\lambda^4}{3} - \frac{32\lambda^3}{3} + 4\lambda^2 + \frac{4\lambda}{3} + \frac{1}{3}\right)^n - \epsilon_n(\lambda) = 0 \tag{5.37}$$

$$\left(4\lambda^2 - 4\lambda + 1\right)^n - \mu_n(\lambda) = 0 \tag{5.38}$$

associated with the inner products of the tensor products of the symmetric tensor fields and the tensor product of the antisymmetric tensor fields respectively. As a result we find both real and imaginary valued solutions. To provide a physical interpretation we consider the real-valued solutions and define the parameter transformation (5.14)

$$\kappa_{\epsilon_n} := \kappa_{\lambda(\epsilon_n)}(\partial_{\epsilon_n}\lambda(\epsilon_n))^2 \tag{5.39}$$

of the quantum Fisher information on the 1-parameter family of Schmidt coefficient decomposed quantum states according to formula (5.14). In this way we find the minumum number $M_\delta(\epsilon_n)$ of measurements

$$M_\delta(\epsilon_n) = \frac{1}{\epsilon_n^2\delta^2}\frac{1}{\kappa_{\epsilon_n}} \tag{5.40}$$

as defined in (5.19) for achieving an estimation with fixed relative error $\delta$. The result is illustrated with $\delta \equiv 1$ for the first five powers in figure 3.

### 5.2.1 Discussion for monotones from the symmetric part

Let us begin to analyze the estimation of the entanglement monotones associated with the inner products on the tensor products of the symmetric tensor

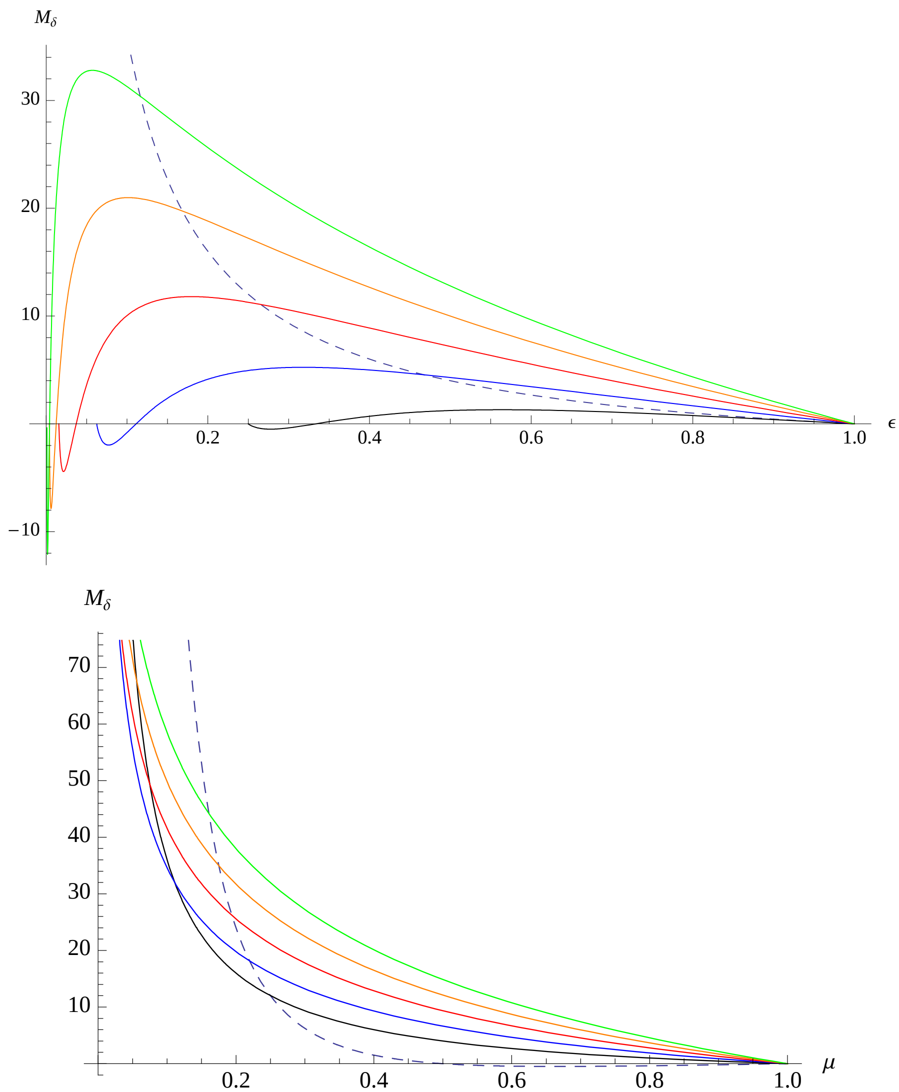



Figure 3: estimation of entanglement monotones $\epsilon_n$ and purity monotones $\mu_n$ for the first five orders (each order corresponds to the same color as used in the plot in figure 1). It shows the number $M_\delta$ of measurements in dependence of the value $\epsilon_n$ (and $\mu_n$ respectively) of the monotones required for achieving an estimation in a $99,9\%$ confidence interval with fixed relative error $\delta \equiv 1$. The dashed curves in the first plot correspond to the estimation of the linear entropy, and in the second plot to the negative-valued estimation of the purity as done in (5.27) when 'reflected' on the $\mu$-axis.

field in the first plot of figure 3. While the estimation of the linear entropy (dashed curve) diverges for weakly entangled states in the limit $\epsilon_n \to 0$ (see section 5.1.2 and [8]), we find for the tensorial monotones an approximative improvement with a finite number of measurements towards the regime of weak entanglement in dependence of the tensor field order $n$. The inflection point into negative values indicates the boundary of the regime where the approximation looses its validity. The validity of the approximation into the regime of weekly entanglement may become enlarged by considering inner products on tensor fields of higher order. The green curve in the first plot of figure 3 corresponds to the highest order example $n = 5$ and clearly illustrates this enlargement when compared to the lower tensor field orders.

### 5.2.2 Discussion for monotones from the anti-symmetric part

The estimation of the purity monotones associated with inner products on the tensor products of the anti-symmetric tensor fields is illustrated in the second plot of figure 3. All curves clearly show here an efficient estimation for weakly entangled states, i.e. for all states close to $\mu_n = 1$. The dashed curve corresponds here to the estimation of the standard purity (section 5.1.3) when reflected on the $\mu$-axis into positive values. Indeed, only the curves associated with the inner products on the tensor fields may admit a physical interpretation.

## 5.3 Conclusions and outlook

In the geometric formulation of quantum mechanics one considers the Fubini-Study metric at the first place. Any action of the symmetry group of entanglement on a family of entangled quantum state vectors induces a family of degenerate pull back tensor fields each defined as pull back of the Fubini-Study metric from the Hilbert space to the Lie group $SU(2) \times SU(2)$. Along the decomposition of the Fubini-Study metric into a Riemannian and a symplectic tensor field one finds a decomposition

$$\kappa_{SU(2)\times SU(2)} = \eta_{SU(2)\times SU(2)} + i\omega_{SU(2)\times SU(2)}$$

into degenerate symmetric and anti-symmetric pullback structures. Via an inner product on higher order tensor fields it is possible to identify two classes of monotonic functions characterizing the entanglement and purity of a bipartite quantum system. These geometrically constructed classes of entanglement and purity monotones provide advantages in the estimation of entangled qubits when compared to standard entanglement and purity monotones. The basic picture emerging here may be subsumed as follows. While the inner product

$$\left\langle \eta^{\otimes n}_{SU(2)\times SU(2)} \middle| \eta^{\otimes n}_{SU(2)\times SU(2)} \right\rangle$$

yields an *approximative* efficient entanglement estimation for *all* state vectors, one finds for the inner product

$$\left\langle \omega^{\otimes n}_{SU(2)\times SU(2)} \middle| \omega^{\otimes n}_{SU(2)\times SU(2)} \right\rangle$$

an *exact* efficient purity estimation for *weakly* entangled state vectors.

It would be interesting to investigate whether this approach admits also advantages in the entanglement estimation of more general composite quantum systems involving multi-partite systems, mixed quantum states and infinite dimensional Hilbert spaces. As a matter of fact, such a generalization becomes directly testable by the tensor field-valued pairing

$$\{\rho_{\vec{\lambda}}\}_{\vec{\lambda}\in\mathcal{M}} \times \{U^{-1}(g)dU(g)^{\otimes k}\}_{g\in\mathcal{G}} \tag{5.41}$$

$$\downarrow \tag{5.42}$$

$$\rho_{\vec{\lambda}}\Big(U^{-1}(g)dU(g)^{\otimes k}\Big) \equiv \kappa_{\mathcal{G}}(\vec{\lambda}) \tag{5.43}$$

as described here in section 4.5 between general manifolds $\{\rho_{\vec{\lambda}}\}_{\vec{\lambda}\in\mathcal{M}}$ of quantum states and *invariant operator valued tensor fields* $U^{-1}(g)dU(g)^{\otimes k}$ on general Lie groups $\mathcal{G}$ associated with a unitary representation $U : \mathcal{G} \to \mathcal{U}(\mathcal{H})$. A generalization in several directions could therefore be tackled by focusing on the corresponding inner products on tensor fields

$$\left\langle \kappa_{\mathcal{G}}^{\otimes n}(\vec{\lambda}) \,\middle|\, \kappa_{\mathcal{G}}^{\otimes n}(\vec{\lambda}) \right\rangle .$$

A deeper understanding on the relation with the quantum Fisher information and all its possible generalized variants (see [21] and references therein) finally, may come along here in terms of a geometrization of the $C^*$-algebraic approach of quantum mechanics [41,61] (see appendix B) including star products of quantum tomograms [79] which may close a circle to the empirical bounds on precision of quantum measurements in terms of generalized uncertainty relations [86].

Indeed, some fundamental aspects in this direction have been found in section 3 by providing a general framework for the translation of any given quantum statistical measure

$$S : D(\mathcal{H}) \to \mathbb{R}_+, \tag{5.44}$$

including the von Neumann entropy, in terms of a functional

$$\tilde{S} : \mathcal{F}_K(\Gamma) \to \mathbb{R}_+ \tag{5.45}$$

on the $C^*$-algebra of Kähler functions

$$f_A(g) := \frac{\langle g| A |g\rangle}{\langle g |g\rangle} = \frac{\langle \psi_0| U(g)^\dagger A U(g) |\psi_0\rangle}{\langle \psi_0 |\psi_0\rangle} \tag{5.46}$$

on a unitary orbit $\Gamma \subset \mathcal{H}_0$. These Kähler functions are deeply connected to Husimi functions

$$f_\rho(g) = \frac{\langle g| \rho |g\rangle}{\langle g |g\rangle} = \frac{\langle \psi_0| U(g)^\dagger \rho U(g) |\psi_0\rangle}{\langle \psi_0 |\psi_0\rangle}$$

$$= \mathrm{Tr}\left(\rho U(g) \frac{|\psi_0\rangle \langle \psi_0|}{\langle \psi_0 |\psi_0\rangle} U(g)^\dagger\right) \equiv \mathrm{Tr}(\rho U(g) E_0 U(g)^\dagger) \tag{5.47}$$

– and therefore to a special subclass of quantum tomograms [79] – and Wehrl type entropy inequalities as shown in section 3.1. A possible foundation for generalized uncertainty relations finally, may become available in terms of geometric inequalities as worked out in section 3.2 for the Robertson Schrödinger inequality and for a version of a quantum Cramér Rao inequality. Actually, the latter has been sufficient for taking into account some of the current developments of quantum estimation theory – in particular – when applied to the empirical bounds of entanglement quantification as considered here in this last section of the underlying work.

At this point we shall remark: The challenge on an efficient empirical quantification of quantum entanglement can be seen 'embedded' according to Pawel Horodecki into a more general set of questions of quantum information theory [80]. Horodecki indicated this class of questions in 2002 as follows:

*"What kind of information can be extracted from an unknown quantum state at small measurement cost?"*

At least from the physicist point of view, this is perhaps one of the most elementary question to begin with, when dealing with the notion of quantum information. What we know from basic quantum mechanics is that we can extract Born type probability measures

$$\mathrm{Tr}(\rho E_x) = p(x) \tag{5.48}$$

or more general, quantum tomograms (compare also formula (5.47))

$$\mathrm{Tr}(\rho U_s E_x U_s^\dagger) \equiv \mathcal{P}(x_s) \tag{5.49}$$

at 'low cost' as they require only one POVM $\{E_x\}_{x\in Q}$ related correspondently to only one single observable. This is in contrast to general quantum statistical measures like the purity

$$\mathrm{Tr}(\rho\rho) \tag{5.50}$$

or the von Neumann entropy

$$\mathrm{Tr}(\rho \ln \rho) \tag{5.51}$$

requiring a tomographic reconstruction of the state $\rho$ involving a possibly continuos set of POVMs related to a complete set of observables.

The content of this work has been focused on a third class of measures being extractable from quantum states in terms of what we could call *geometric measures*

$$\mathrm{Tr}(d\rho \otimes d\rho) \tag{5.52}$$

$$\mathrm{Tr}(\rho U^\dagger dU^{\otimes k}) \tag{5.53}$$

provided by the geometric formulation of quantum mechanics.

In fact, the geometric formulation of quantum mechanics provides perhaps one of the most appropriated frameworks to tackle Horodecki's question as stated above. To be specific, the Fubini Study metric provides a conceptual unification of the following both concepts, that is

- information of a quantum state, *and*

- measurement costs

on the one hand by quantifying entanglement – as an extractable information from a quantum state – in terms of

- the pullback on the symmetry group of entanglement,

and on the other hand, by quantifying the minimum of measurement costs with

- the pullback on a 1-parameter family of quantum state vectors.

Of course it may remain an open problem to identify all properties of a general quantum state (and therefore the 'complete' information from an unknown quantum state) at small measurement cost. Geometry indeed, will always provide the most fundamental mathematical language for describing the state attributes of our world – no matter if classical or quantum.

# A Weyl Systems

Consider a symplectic vector space $(V, \omega)$ with an Abelian non-compact Lie group of translations $V \cong \mathbb{R}^{2n}$ endowed with a symplectic structure $\omega$. Any generating vector field of translations in the associated Lie algebra may therefore be identified with an element $v \in V$ according to the isomorphism $T_v V \cong V$.
We consider an irreducible unitary (projective) representation of $V$ on a Hilbert space $\mathcal{H}$ defined by a *Weyl-system*, that is, a map $W : V \to \mathcal{U}(\mathcal{H})$, satisfying the following conditions [66]:

1. $W$ is strongly continuous as a function on $V$;
2. $W(v + v') = e^{-\frac{i}{2}\omega(v,v')} W(v) W(v')$.

According to the Stone-von Neumann theorem it is possible to write

$$W(v) = e^{iR(v)}, \tag{A.1}$$

in terms of a self-adjoint realization $R(v)$ of a generating vector field of translations $v$. It implies

$$[R(v), R(v')] = -i\omega(v, v'). \tag{A.2}$$

Hence, all self-adjoint operators $R(v)$ amounted with elements of a Lagrangian subspace of $V$ commute.

In this regard we recall some basic notions of symplectic geometry (see e.g. [87] p. 403) as follows. To any subspace

$$U \subset V, \tag{A.3}$$

there is a $\omega$-*orthogonal complement*

$$U^{\perp} := \{v \in V | \omega(v, u) = 0, u \in U\}. \tag{A.4}$$

The subspace $U$ may define either a *isotropic* or a *coisotropic* subspace if $U \subset U^{\perp}$ or $U^{\perp} \subset U$ respectively. An isotropic subspace $U$ which induces together with another isotropic subspace $U'$ a splitting

$$V = U \oplus U', \tag{A.5}$$

is called *Lagrangian* subspace. In turns out that $U$ is Lagrangian iff $U$ is isotropic and has $\mathrm{Dim}(U) = \frac{1}{2}\mathrm{Dim}(V)$, which is equivalent with $U = U^{\perp}$.
A given Lagrangian subspace $U$ and its associated splitting into Lagrangian subspaces

$$v := (u, f) \in U \oplus U' \tag{A.6}$$

induces a decomposition on the Weyl system into

$$W(v) = W(u, f) = e^{-\frac{i}{2}\omega((u,0),(0,f))}U(u)V(f) \tag{A.7}$$

with

$$U \equiv W|_U : U \to \mathcal{U}(\mathcal{H}) = \mathcal{U}(L^2(U)) \tag{A.8}$$

and

$$V \equiv W|_{U'} : U' \to \mathcal{U}(\mathcal{H}) = \mathcal{U}(L^2(U)). \tag{A.9}$$

For $\psi \in L^2(U)$ the Weyl system action reads [66]

$$U(U')\psi(u) = \psi(u + U') \tag{A.10}$$

$$V(f)\psi(u) = e^{-i\omega((u,0),(0,f))}\psi(u). \tag{A.11}$$

## A.1 Introducing canonical coordinates

### A.1.1 Real coordinates

We have seen that given a symplectic structure $\omega$ on $V$, we may consider a splitting into Lagrangian subspaces. In this way we may introduce canonical coordinates

$$v = (q, p) \in \mathbb{R}^n \oplus (\mathbb{R}^n)^* \tag{A.12}$$

according to the splitting associated to the symplectic structure

$$\omega = \delta_{jk} dp^j \wedge dq^k. \tag{A.13}$$

Actually, a vector space of translations $V$ may admit different symplectic structures and associated splittings into Lagrangian subspaces in dependence of the physical and experimental setting. For instance, we may deal either with a composite phase space of $n$ particles in one dimension or with one particle in $n$ dimensions:

$$V \cong \mathbb{R}^{2n} \cong \begin{cases} (T^*\mathbb{R})^n \\ T^*\mathbb{R}^n \end{cases} \tag{A.14}$$

Correspondently, these cases may endowed with distinguished symplectic structures

$$\omega := \begin{cases} \bigoplus_{j=1}^n \omega_j \text{ with } \omega_j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \\ \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}. \end{cases} \tag{A.15}$$

Let $\{X_j\}_{j \in J}$ be a basis on $V \cong \mathbb{R}^{2n}$ (or a basis of generating vector fields in the Lie algebra of $V$ respectively). Such a basis becomes therefore represented within a Weyl system by a set of Hermitian operators $\{R(X_j)\}_{j \in J}$ on $\mathcal{H}$ with different position and momentum operator realizations

$$\{R(X_1), R(X_2), ..., R(X_{n+1}), R(X_{n+2}), .., R(X_{2n})\} := \begin{cases} \{Q_1, P_1, ..., Q_n, P_n\} \\ \{Q_1, Q_2, ..., P_{n-1}, P_n\} \end{cases} \tag{A.16}$$

in dependence of the symplectic structure.

**Remark A.1.** *Both settings may be related in terms of a permutation matrix [59].*

In any of the above two splittings in real canonical coordinates, we find a Weyl system

$$W(q,p) = e^{i(q^j P_j - p^j Q_j)} \tag{A.17}$$

### A.1.2 Complex coordinates

The latter expression becomes identified with the *displacement operator* when rewritten in complex coordinates

$$W(z) \equiv e^{z^j a_j^\dagger - \bar{z}^j a_j}, \tag{A.18}$$

as being used in quantum optics for $n$ bosons in one dimension (see e.g. [59]). The relation to real coordinates is given by taking into the decomposition

$$z^j = \frac{1}{\sqrt{2}}(q^j + ip^j) \tag{A.19}$$

in real and imaginary coordinates. Indeed, by setting $\kappa_1 = \frac{1}{\sqrt{2}}$, both descriptions become equivalent, i.e.

$$W(z) = W(q,p). \tag{A.20}$$

The Weyl system generators $iR(X_j)$ are related to the basis

$$\{Z_j\}_{j\in J} := \{X_j + iX_{n+j}\}_{j\in J} \tag{A.21}$$

on $V \cong \mathbb{C}^n \cong \mathbb{R}^{2n}$ yielding the mode operators

$$R(Z_j) = \kappa_1(Q_j + iP_j) := a_j, \quad R(\bar{Z}_j) = \kappa_1(Q_j - iP_j) := a_j^\dagger, \tag{A.22}$$

with $\kappa_1 \in \mathbb{R}$.

## A.2 Symplectic transformation

Let us focus on those transformations which leave the underlying symplectic structure of a given Weyl-system invariant. These transformation define the Lie group $Sp(2n, \mathbb{R})$ of symplectic transformation

$$S : V \to V, \quad \omega(Sv, Sv') = \omega(v, v'). \tag{A.23}$$

On the level of the Weyl-system we find, that the following diagram

$$\begin{array}{ccc} (V, \omega) & \xrightarrow{W} & \mathcal{U}(\mathcal{H}) \\ {\scriptstyle Sp(2n,\mathbb{R})\ni S}\Big\downarrow & & \Big\downarrow{\scriptstyle \Phi_{U_S}\in Aut(\mathcal{U}(\mathcal{H}))} \\ (V, \omega) & \xrightarrow{W_S} & \mathcal{U}(\mathcal{H}), \end{array} \tag{A.24}$$

commutes, if we identify $\Phi_{U_S}$ with an automorphism

$$\Phi_{U_S}(W(v)) = U_S^{-1} W(v) U_S = W(Sv) := W_S(v) \tag{A.25}$$

with $U_S \in U(\mathcal{H})$. This induces a transformation on the infinitesimal level of the Hermitian operator representation given by

$$R_S(v) \equiv R(Sv) = U_S^{-1} R(v) U_S. \tag{A.26}$$

The application of a symplectic transformation on a translation is taken into account in the affine symplectic group

$$ISp(2n, \mathbb{R}) := Sp(2n, \mathbb{R}) \vec{\times} \mathbb{R}^{2n}. \tag{A.27}$$

Many quantum dynamical systems, as those being considered in quantum optics and condensate matter, are based on unitary representations of the latter group, defining in this way the so-called metaplectic representations

[59]. Such representations may be generated by Hamiltonians

$$H = \underbrace{\sum_{k}^{n} g_k^{(1)} a_k^\dagger}_{\text{translations}} + \underbrace{\sum_{k,l=1}^{n} g_{kl}^{(3)} a_k^\dagger a_l^\dagger}_{\text{squeezing}} + \underbrace{\sum_{k>l=1}^{n} g_{kl}^{(2)} a_k^\dagger a_l}_{\text{mixing}} \tag{A.28}$$

that are linear and bilinear in $n$ field modes. They induce translations, squeezing and mixing operations associated to unitary representations of non-compact and compact subgroups of $ISp(2n, \mathbb{R})$. In the case $n = 2$, for instance, we encounter unitary representations of the subgroups

$$\mathbb{R}^4, \quad SU(1,1) \text{ and } SU(2) \tag{A.29}$$

respectively.

## A.3 The Wigner-Weyl correspondence

Given a Weyl system $W : V \to \mathcal{U}(\mathcal{H})$ we may define the so-called *characteristic function* of a generic operator $A$ on $\mathcal{H}$ by

$$\chi[A](v) := \mathrm{Tr}(AW(v)). \tag{A.30}$$

Vice versa, to a function in $L^1(\mathbb{R}^{2n}, d^n\alpha d^n x)$ one may associate an Hilbert-Schmidt operator on $L^2(\mathbb{R}^n, d^n x)$ as follows [79]: Consider the Fourier transform $\tilde{f}$ within the inverse Fourier transform formula

$$f(q,p) = \int d^n\alpha d^n x \tilde{f}(\alpha, x) e^{i(\alpha q - xp)} \tag{A.31}$$

and replace the 1-dimensional irreducible unitary representation

$$e^{i(\alpha q - xp)} \tag{A.32}$$

of $V \cong \mathbb{R}^{2n}$ with the *projective* irreducible unitary representation of $V$, defined by the corresponding Weyl-system

$$W(x,\alpha) := e^{i(\alpha Q - xP)}, \tag{A.33}$$

yielding the operator

$$W(f) := \int d^n\alpha\, d^n x\, \tilde{f}(\alpha, x) W(x,\alpha). \tag{A.34}$$

This formula may now be compared with the reconstruction formula [59]

$$A = \int_V \frac{d^{2n}v}{\pi^n} \chi[A](v) W^\dagger(v) \tag{A.35}$$

of a generic operator $A$ from its characteristic function. At this point we may introduce the so-called *Wigner function* of an operator $A$ as the Fourier transform of the characteristic function according to

$$\mathcal{W}[A](q,p) := \int d^n\alpha\, d^n x\, \chi[A](\alpha, x) e^{i(\alpha q - xp)}. \tag{A.36}$$

In conclusion, the Weyl system framework provides a fundamental mathematic tool to relate operators to functions and vice versa. In particular, this allows to consider a Quantization-Dequantization algorithm scheme involving both Weyl systems $W$ and Wigner functions $\mathcal{W}$ according to the following table:

| | Quantization | Dequantization |
|---|---|---|
| 'Input' | $f$ | $A$ |
| 'Output' | $W(f)$ | $\mathcal{W}[A]$ |

The outlined dequantization correspondence should be taken with great care to avoid physical misinterpretations. In particular, the Wigner function of quantum state may also take negative values providing thus not a probability but a quasi-probability distribution on the symplectic space $V$. Probability distributions may be defined on a Lagrangian subspace in terms of *quantum tomograms* in terms of a inverse Radon transform of the Wigner function [79]. Vice versa, the Wigner function of a quantum state may be reconstructed from a given a family of quantum tomograms in terms of a Radon transform. Quantum tomograms are therefore of fundamental importance in quantum estimation theory [83].

# B Tensor fields from geometrized $C^*$-algebras

By evaluating a quantum state on invariant operator valued tensor field on a Lie group $\mathcal{G}$ as done in section 4.5, we may identify the resulting covariant classical tensor field as a pull-back induced by the projection of the Lie group $\mathcal{G}$ on a homogenous space $\mathcal{G}/\mathcal{G}_0$ generated by the associated action of $\mathcal{G}$ in the vector space $\mathcal{A}$ of a $C^*$-algebra. This suggests consider a more general class of tensor fields on $\mathcal{G}$ by taking into account tensor fields from geometrized $C^*$-algebras and their pull-back to the group.

Let us restart for this purpose by considering the definition of a finite dimensional $C^*$-algebra at the first place:

**Definition B.1** (Finite dimensional $C^*$ algebra)**.** *A Banach space $\mathcal{A}$ which is endowed with an associative product*

$$\mathcal{A} \times \mathcal{A} \to \mathcal{A} \tag{B.1}$$

$$(A, B) \mapsto A \cdot_K B, \tag{B.2}$$

*and an involution operation*

$$\mathcal{A} \to \mathcal{A} \tag{B.3}$$

$$A \mapsto A^*, \tag{B.4}$$

*which is compatible with the Banach norm according to*

$$||AA^*|| = ||A||^2, \tag{B.5}$$

*is called a finite dimensional $C^*$-algebra.*

Any finite dimensional $C^*$-algebra is isomorphic to an algebra $M_n(\mathbb{C})$ of complex $n \times n$-matrices. Here we note that the usual row-by-column matrix product

$$(A, B) \mapsto AB \tag{B.6}$$

is a special example of an associative product. In general we may deal in this regard also with alternative products not necessarily coinciding with the usual matrix product. For instance we may use

$$A \cdot_K B \equiv AKB \tag{B.7}$$

with a fixed matrix $K \in M_n(\mathbb{C})$, which satisfies

$$(A \cdot_K B) \cdot_K C = A \cdot_K (B \cdot_K C), \tag{B.8}$$

for all $A, B, C \in \mathcal{A}$.
By considering a realification of the the $C^*$-algebra $M_n(\mathbb{C})$ from $\mathcal{A} \cong \mathbb{R}^{n^2} \oplus i\mathbb{R}^{n^2}$ to

$$\mathcal{A}_{\mathbb{R}} \cong \mathbb{R}^{2n^2}, \tag{B.9}$$

we may deal with the question whether the later stands in an analogy with a 'classical phase space' manifold, admitting classical tensors in a similar way it has been constructed on the realification of a Hilbert space in [1]. In contrast to a Hilbert space, we have to take new structures into account which come along the definition of a $C^*$-algebra. They essentially consist of

- an associative bi-linear product, which is non-commutative and
- a Banach Norm, compatible with the product.

As we will see in the following, both points provide implications for the construction of tensor fields, in particular defined in contra-variant form on the dual vector space $\mathcal{A}^*$, resp. in covariant form on the vector space of the $C^*$-algebra $\mathcal{A}$. Let us start with the construction of contra-variant tensors by following the argumentation line as proposed in [61].

## B.1 Contra-variant tensors on the dual vector space $\mathcal{A}^*$

Consider the dual space

$$\mathcal{A}^* := \mathrm{Lin}(\mathcal{A}) \tag{B.10}$$

of complex valued linear functionals

$$\alpha : \mathcal{A} \to \mathbb{C} \tag{B.11}$$

on $\mathcal{A}$. Note that the duality isomorphism is considered here in the category of (finite dimensional) vector spaces and not in the category of abstract $C^*$-algebras. Here we observe that for all elements $A \in \mathcal{A}$ there exits an element $\widehat{A}$ in the bi-dual $(\mathcal{A}^*)^* := \mathrm{Lin}(\mathcal{A}^*)$, i.e. the space of linear functionals

$$\widehat{A} : \mathcal{A}^* \to \mathbb{C} \tag{B.12}$$

on the dual space $\mathcal{A}^*$, provided by

$$\widehat{A}(\alpha) := \alpha(A) \tag{B.13}$$

for all $\alpha \in \mathcal{A}^*$. The relation (B.13) defines an embedding

$$\mathcal{A} \hookrightarrow \mathrm{Lin}(\mathcal{A}^*) \subset \mathcal{F}(\mathcal{A}^*) \tag{B.14}$$

$$A \mapsto \widehat{A}, \tag{B.15}$$

of $\mathcal{A}$ into to the bi-dual $(\mathcal{A}^*)^*$ and will be of fundamental importance for the following discussion. In particular, within the bi-dual space of linear functionals on $\mathcal{A}^*$ we are allowed to identify a commutative product among linear 'functions' which provides a quadratic function

$$(\widehat{A} \cdot \widehat{B})(\alpha) := \widehat{A}(\alpha) \cdot \widehat{B}(\alpha), \tag{B.16}$$

and therefore caries the usual differential calculus. In particular we will have an exterior derivative on elements of $\mathrm{Lin}(\mathcal{A}^*)$ where the usual Leibniz-rule is available according to

$$d(\widehat{A}\cdot\widehat{B}) = d\widehat{A}\cdot\widehat{B} + \widehat{A}\cdot d\widehat{B}. \tag{B.17}$$

In this way we may identify $d\widehat{A}$ as a 1-form on $\mathcal{A}^*$, resp. as a section in the global trivialized co-tangent bundle over the dual space according to

$$d\widehat{A} : \mathcal{A}^* \to T^*\mathcal{A}^* = \mathcal{A}^* \times (\mathcal{A}^*)^* \equiv \mathcal{A}^* \times \mathcal{A}. \tag{B.18}$$

$$\alpha \mapsto (\alpha, \widehat{A}) \equiv (\alpha, A), \tag{B.19}$$

i.e. a section in a vector bundle $\mathcal{A}^* \times \mathcal{A}$.
On the other hand we may translate the non-commutative product

$$A \cdot_K B \tag{B.20}$$

on $\mathcal{A}$ into a corresponding non-commutative product on $\mathrm{Lin}(\mathcal{A}^*)$, whenever we consider the embedding (B.13), as a way to translate the product between operators to a product between functions according to

$$\alpha(A \cdot_K B) =_{(B.13)} \widehat{A \cdot_K B}(\alpha) := \widehat{A}(\alpha) \star_K \widehat{B}(\alpha). \tag{B.21}$$

Such a product appears in the literature of non-commutative geometry and provides an instance of a so called *star-product* (see e.g. [88]). It is non-local product since it is associated with row-by-column product which is the analog of the *convolution product*[16]. The non-commutative structure of $\mathcal{A}^*$ is now completely encoded in the product $\star_K$ between these functions in the bi-dual space $\mathrm{Lin}(\mathcal{A}^*) \subset \mathcal{F}(\mathcal{A}^*)$.

[16]For instance the usual Moyal product is the convolution product on the Heisenberg-Weyl group.

This essentially allows us now to identify a contra-variant rank-2 tensor field on $\mathcal{A}^*$ according to the construction

$$\tau_K(d\widehat{A}, d\widehat{B})(\alpha) := \alpha(A \cdot_K B) = \widehat{A}(\alpha) \star_K \widehat{B}(\alpha). \tag{B.22}$$

Hence, we arrived to a fundamental relation which provides us a *geometrization* of the dual space of any given finite dimensional $C^*$-algebra [61]. Moreover, we may consider the decomposition of the associative product on the $C^*$-algebra according to

$$A \cdot_K B = \frac{1}{2}(A \cdot_K B + B \cdot_K A) + \frac{1}{2}(A \cdot_K B - B \cdot_K A) \tag{B.23}$$

and identify a corresponding induced decomposition of the tensor field by

$$\tau_K(d\widehat{A}, d\widehat{B})(\alpha) = \alpha(A \circ_K B) + i\alpha([A, B]_K), \tag{B.24}$$

which geometrizes the symmetric and the anti-symmetric products

$$A \circ_K B := \frac{1}{2}(A \cdot_K B + B \cdot_K A) \tag{B.25}$$

$$[A, B]_K := \frac{1}{2i}(A \cdot_K B - B \cdot_K A) \tag{B.26}$$

by means of a real symmetric

$$R_K(d\widehat{A}, d\widehat{B})(\alpha) := \alpha(A \circ_K B), \tag{B.27}$$

and an imaginary anti-symmetric

$$\Lambda_K(d\widehat{A}, d\widehat{B})(\alpha) := \alpha([A, B]_K), \tag{B.28}$$

(2,0)-tensor field on $\mathcal{A}^*$. Both structures are equivalently defined via symmetrized and anti-symmetrized star-products on the bi-dual of linear functionals on $\mathcal{A}^*$ according to

$$R_K(d\widehat{A}, d\widehat{B})(\alpha) = \frac{1}{2}(\widehat{A} \star_K \widehat{B} + \widehat{B} \star_K \widehat{A})|_\alpha := (\widehat{A}, \widehat{B})_k|_\alpha \tag{B.29}$$

$$\Lambda_K(d\widehat{A}, d\widehat{B})(\alpha) = \frac{1}{2i}(\widehat{A} \star_K \widehat{B} - \widehat{B} \star_K \widehat{A})|_\alpha := \{\widehat{A}, \widehat{B}\}_k|_\alpha. \tag{B.30}$$

It is clear that distinguished bi-linear products $A \circ_K B$ and $[A, B]_K$ and associated contra-variant tensors may be realized in this regard in dependence of the choice of the real matrix $K \in M_n(\mathbb{C})$ coming along the product

$$A \cdot_K B \equiv AKB \tag{B.31}$$

of the $C^*$-algebra. In particular, if we choose the identity

$$K \equiv \mathbb{1} \in M_n(\mathbb{C}) \tag{B.32}$$

we recover the standard non-commutative associative matrix product, which allows to identify

$$R_{\mathbb{1}}(d\widehat{A}, d\widehat{B})(\alpha) := \alpha([A, B]_+) = (\widehat{A}, \widehat{B})|_\alpha, \tag{B.33}$$

$$\Lambda_{\mathbb{1}}(d\widehat{A}, d\widehat{B})(\alpha) := \alpha([A, B]_-) = \{\widehat{A}, \widehat{B}\}|_\alpha, \tag{B.34}$$

a Riemanian and a Poisson tensor field, which are induced by a Jordan and a Lie product respectively. Correspondently, theses products provide a Jordan and a Poisson bracket on the bi-dual $\mathrm{Lin}(\mathcal{A}^*) \subset \mathcal{F}(\mathcal{A}^*)$ of linear functionals on $\mathcal{A}^*$. Moreover, by restricting to the real elements of a complex matrix algebra endowed with the standard matrix product corresponding to $K = \mathbb{1}$, we may find that both products, the symmetric Jordan product and and Lie product are closed in respect to a Lie-Jordan algebra [41]. Hence, we may also directly start on the subspace of real elements provided by the Hermitian matrices and find an equivalent geometrization on the space of observables in terms of a Riemanian and a Poisson structure

$$R_{\mathbb{1}}(d\widehat{A}, d\widehat{B})(\xi) := \xi([A,B]_+) = (\widehat{A}, \widehat{B})|_\xi, \tag{B.35}$$

$$\Lambda_{\mathbb{1}}(d\widehat{A}, d\widehat{B})(\xi) := \xi([A,B]_-) = \{\widehat{A}, \widehat{B}\}|_\xi, \tag{B.36}$$

for all

$$\xi \in u^*(n) := \mathrm{Lin}(u(n)) \tag{B.37}$$

and

$$A, B \in u(n), \quad \widehat{A}, \widehat{B} \in \mathrm{Lin}(u^*(n)) \tag{B.38}$$

on the space $u^*(n)$ of real valued linear functionals defined on $u(n)$.

**Remark B.2.** *The geometric structures in (B.35),(B.36) can be recovered by a momentum-map push-forward of a contravariant Hermitian tensor field from a Hilbert space $\mathcal{H} \cong \mathbb{C}^n$ to the corresponding space of observables $u^*(n)$ [43]. In particular, one encounters in this way a Hermitian realization, which provides a generalization of symplectic realizations from symplectic to Poisson manifolds [61].*

We may now 'close a circle', namely by defining a *covariant* defined tensor field on the vector space of a $C^*$-algebra $\mathcal{A}$. For a general state $\omega \in \mathcal{A}^*$ this covariant tensor field will be a degenerate pull-back tensor field from a GNS-constructed Hilbert space $\mathcal{H}_\omega := \mathcal{A}/J_\omega$. The identification of such a covariant tensor field can be made explicit as follows.

## B.2 Covariant tensors on the vector space $\mathcal{A}$

As we remarked at the beginning, a $C^*$-algebra may admit a Banach Norm, which is not necessarily induced by an (Hermitian) inner product. However, from physical point of view, we are interested in the Hilbert spaces identified via the Gelfand-Naimark-Segal-construction (GNS). In this regard we shall restrict our attention to inner products defined by

$$\langle A \, | A \rangle_\omega := \omega(A^\dagger A) \tag{B.39}$$

on $\mathcal{A}$, with

$$\omega \in \mathcal{A}^*, \tag{B.40}$$

a fixed linear functional on $\mathcal{A}$. The latter is identified with a *state*, whenever we associate to it the two additional properties

$$\text{Positivity} :\Leftrightarrow \ \omega(A^\dagger A) > 0 \tag{B.41}$$

$$\text{Normalization} :\Leftrightarrow \ \omega(\mathbb{1}) = \mathrm{Tr}(\rho_\omega) = 1. \tag{B.42}$$

We will denote the space of states associated to a given $C^*$-algebra by $D(\mathcal{A})$. At this point we may consider a *covariant* tensor field on $\mathcal{A}$

$$\tau_\omega(X_A, X_B)(a) := \omega(X_A(a) \cdot X_B(a)), \tag{B.43}$$

defined by a contraction with vector fields

$$X_C : \mathcal{A} \to T\mathcal{A} = \mathcal{A} \times \mathcal{A} \tag{B.44}$$

$$a \mapsto (a, C). \tag{B.45}$$

Hence, we arrive to an invariant (0, 2)-tensor field

$$\tau_\omega(X_A, X_B)(a) = \omega(A^\dagger B). \tag{B.46}$$

A coordinate description is provided as follows. To a given element $A \in \mathcal{A} = M_n(\mathbb{C})$, we may either find a coordinate matrix description

$$A = \sum_{j,k} A_{jk} \left| j \right\rangle \left\langle k \right| . \tag{B.47}$$

or, by considering a basis $\{|e_j\rangle\}_{j\in J}$ on the vector space $\mathcal{A}$, we may expand any given element $A \in \mathcal{A}$ as a vector

$$A \equiv |A\rangle = \sum_{j=1}^{n} \langle e_j \, |A\rangle \, |e_j\rangle = \sum_{j=1}^{n} \alpha^j \, |e_j\rangle \, , \tag{B.48}$$

where we identified coordinate functions

$$\langle e_j \, |A\rangle := \alpha^j(A), \qquad \alpha^j \in \mathcal{A}^*. \tag{B.49}$$

with linear functionals on $\mathcal{A}$. The identification

$$A \equiv |A\rangle \tag{B.50}$$

between (matrix) column-row objects $A$ and (vector) column objects $|A\rangle$ can be made clear due to the 1-to-1 corespondence between the linear functionals and matrix coefficients $A_{jk}$

$$\{\alpha_l\}_{l\in I} \leftrightarrow \{A_{jk}\}_{j,k\in J}, \tag{B.51}$$

resp. basis vectors

$$\{|e_l\rangle\}_{l\in I} \leftrightarrow \{|j\rangle \, \langle k|\}_{j,k\in J}, \tag{B.52}$$

Here we may introduce the notation

$$d\,|A\rangle \equiv |dA\rangle = dA, \tag{B.53}$$

defined by the exterior derivative on the coordinate functions according to

$$|dA\rangle := \sum_{j=1}^{n} d\alpha^j \, |e_j\rangle \tag{B.54}$$

and find

**Proposition B.3.**

$$\tau_\omega \equiv \langle dA \otimes dA \rangle_\omega := \omega(dA^\dagger \otimes dA). \tag{B.55}$$

*Proof.* One has to show that

$$\omega(dA^\dagger \otimes dA)(X_B, X_C)(a) = \omega(B^\dagger C). \tag{B.56}$$

By writing the left hand side in coordinates we find

$$\omega(dA^\dagger \otimes dA)(X_B, X_C)(a) = \sum_{j,k=1}^{n} \omega(|e_j\rangle \langle e_k|) d\alpha^j \otimes d\alpha^k (X_B, X_C)(a) \tag{B.57}$$

with $X_B : a \mapsto (a, B)$ and $X_C : a \mapsto (a, C)$, and their expansion

$$X_B = \sum_{j=1} \langle e_j \,|B\rangle \frac{\partial}{\partial \alpha^j} = \sum_{j=1} \alpha_j(B) \frac{\partial}{\partial \alpha^j} \tag{B.58}$$

$$X_C = \sum_{j=1} \langle e_j \,|c\rangle \frac{\partial}{\partial \alpha^j} = \sum_{j=1} \alpha_j(C) \frac{\partial}{\partial \alpha^j} \tag{B.59}$$

in a holonomic basis of $\mathcal{A}$.

□

To this point, we may focus on some direct implications of the geometrization

$$(\mathcal{A}, \langle A \,|A\rangle_\omega) \to (\mathcal{A}, \langle dA \otimes dA \rangle_\omega) \tag{B.60}$$

of (degenerate) inner products on a given $C^*$-algebra. As indicated before, such a construction stands in a neat relation to the geometrization

$$(\mathcal{H}_\omega, \langle \psi \,|\psi\rangle) \to (\mathcal{H}_\omega, \langle d\psi \otimes d\psi \rangle) \tag{B.61}$$

of the Hermitian inner product $\langle \psi \,|\psi\rangle$ on a Hilbert space by means of a Hermitian tensor field $\langle d\psi \otimes d\psi \rangle$, which we has been considered in the geometric formulation of quantum mechanics as presented here in this work in section 2. In fact, the later Hermitian tensor field may now be recovered from the GNS-construction associated to the above considered geometrized $C^*$-algebra and goes along the following lines.

**Remark B.4.** *In general, the inner product of a vector space $V$ admits a geometrization by means of a covariant tensor field, while the dual $V^*$ takes into account a contra-variant tensor field. The case where $V$ coincides with a Hilbert manifold is described in section 2.2.*

## B.3 Kählerian manifolds from states

We may now consider real coordinate description of the pair $(\mathcal{A}, \langle dA \otimes dA\rangle_\omega)$. For this purpose we find on each element $A \in \mathcal{A}$ the decomposition into

$$A = \sum_{j,k} A_{jk}(A)\,|j\rangle\langle k| \equiv \sum_{j,k}(Q_{jk}(A) + iP_{jk}(A))\,|j\rangle\langle k| \tag{B.62}$$

where $Q_{jk}, P_{jk} \in \mathcal{A}^*$ denote real scalar valued coordinate functions

$$Q_{jk}(A) = \frac{1}{2}\mathrm{Tr}(\widehat{A}(\alpha)+(\widehat{A}(\alpha))^\dagger)\,|j\rangle\langle k|) = \frac{1}{2}\mathrm{Tr}(\alpha(A)+(\alpha(A))^\dagger\,|j\rangle\langle k|) \tag{B.63}$$

$$P_{jk}(A) = \frac{1}{2i}\mathrm{Tr}(\widehat{A}(\alpha) - (\widehat{A}(\alpha))^\dagger)\,|j\rangle\langle k|) = \frac{1}{2i}\mathrm{Tr}(\alpha(A) - (\alpha(A))^\dagger\,|j\rangle\langle k|) \tag{B.64}$$

on $\mathcal{A}$. In this coordinates one finds the 1-forms $dQ_{jk}, dP_{jk} \in \mathcal{T}_1^0(\mathcal{A})$ and an operator valued tensor

$$dA^\dagger \otimes dA = (dQ_{kj} - idP_{kj})\,|j\rangle\langle k| \otimes (dQ_{ls} + idP_{ls})\,|l\rangle\langle s|$$

$$= |j\rangle\langle k\,|l\rangle\langle s|\,(dQ_{kj} - idP_{kj}) \otimes (dQ_{ls} + idP_{ls})$$

$$= |j\rangle\langle s|\,(dQ_{kj} - idP_{kj}) \otimes (dQ_{ks} + idP_{ks})$$

$$= |j\rangle\langle s|\,(dQ_{kj} \otimes dQ_{ks} + dP_{kj} \otimes dP_{ks} + idQ_{kj} \otimes dP_{ks} - idP_{kj} \otimes dQ_{ks}) \tag{B.65}$$

with the tensor product $\otimes$ defined on $\mathcal{T}_1^0(\mathcal{A})$. Hence, the tensor field (B.55), $\langle dA \otimes dA \rangle_\omega$ on $\mathcal{A}$ will read in these coordinates as

$$\mathrm{Tr}(\omega\,|j\rangle\,\langle s|)(dQ_{kj}\otimes dQ_{ks}+dP_{kj}\otimes dP_{ks}+idQ_{kj}\otimes dP_{ks}-idP_{kj}\otimes dQ_{ks}). \quad \text{(B.66)}$$

### B.3.1 Pure states

In this setting we identify a GNS-construction map

$$\pi_\omega : \mathcal{A} \to \mathcal{H}_\omega := \mathcal{A}/J_\omega \qquad \text{(B.67)}$$

defined by the Gelfand ideal $J_\omega$. If we consider the case of a pure state $\omega := |1\rangle\,\langle 1|$ one gets due to the coefficients in (B.66),

$$\mathrm{Tr}(|1\rangle\,\langle 1\,|j\rangle\,\langle s|) = \delta_{s1}\delta_{j1}, \qquad \text{(B.68)}$$

a *degenerate* covariant tensor field on $\mathcal{A}$

$$dQ_{k1}\otimes dQ_{k1} + dP_{k1}\otimes dP_{k1} + idQ_{k1}\otimes dP_{k1} - idP_{k1}\otimes dQ_{k1}, \qquad \text{(B.69)}$$

which is a $\pi_\omega$-induced pull-back from a Kählerian tensor field living on the 'quotient' $\mathcal{H}_\omega$.This can be seen by setting $Q_{k1} := q_k, P_{k1} := p_k$ which relates the degenerate pull-back tensor (B.69) to a Riemannian and symplectic structure

$$\begin{aligned} &dq_k\otimes dq_k + dp_k\otimes dp_k + idq_k\otimes dp_k - idp_k\otimes dq_k \\ &= dq_k\odot dq_k + dp_k\odot dp_k + idq_k\wedge dp_k \end{aligned} \qquad \text{(B.70)}$$

on the GNS-constructed Hilbert space $\mathcal{H}_\omega = \mathbb{C}^n$.

### B.3.2 Mixed states

Let us consider next the generalized case of a mixed state $\omega := \sum_{i=1}^m \lambda_i\,|i\rangle\,\langle i|$ in terms of a rank-m projector with $\lambda_i > 1$ and $\sum_{i=1}^m \lambda_i = 1$. The coefficients in (B.66) become then

$$\sum_{i=1}^{m} \mathrm{Tr}(\lambda_i |i\rangle \langle i |j\rangle \langle s|) = \sum_{i=1}^{m} \lambda_i \delta_{is}\delta_{ij}. \tag{B.71}$$

In this way one finds the degenerate covariant tensor field on $\mathcal{A}$

$$\begin{aligned}
&\sum_{i=1}^{m} \lambda_i (dQ_{ki} \otimes dQ_{ki} + dP_{ki} \otimes dP_{ki} + idQ_{ki} \otimes dP_{ki} - idP_{ki} \otimes dQ_{ki}) \\
&= \sum_{i=1}^{m} dq_k^{(i)} \otimes dq_k^{(i)} + dp_k^{(i)} \otimes dp_k^{(i)} + idq_k^{(i)} \otimes dp_k^{(i)} - idp_k^{(i)} \otimes dq_k^{(i)} \\
&= \sum_{i=1}^{m} dq_k^{(i)} \odot dq_k^{(i)} + dp_k^{(i)} \odot dp_k^{(i)} + idq_k^{(i)} \wedge dp_k^{(i)},
\end{aligned} \tag{B.72}$$

with $\lambda_i Q_{ki} := q_k^{(i)}, \lambda_i P_{ki} := p_k^{(i)}$. Again, we may identify this structure with a pull back tensor field from the corresponding GNS-constructed Hilbert space $\mathcal{H}_\omega = \bigoplus_{i=1}^{m} \mathbb{C}^n$.

### B.3.3 Maximal rank states

It becomes clear that if we consider a (normalizing) multiple of the identity

$$\omega = \frac{1}{n}\mathbb{1}, \quad c \in \mathbb{R}_0 \tag{B.73}$$

we may identify an Hermitian inner product

$$\langle A |A\rangle_{\frac{1}{n}\mathbb{1}} := \frac{1}{n}\mathrm{Tr}(A^\dagger A), \tag{B.74}$$

which makes out of any vector space of a given $C^*$-algebra $\mathcal{A} \cong M_n(\mathbb{C})$ a Hilbert space. This inner product may admit two possible interpretations. First it recovers the inner product which induces the Hilbert-Schmidt norm on a finite dimensional Hilbert space of 'Hilbert Schmidt operators', whenever we encounter the isomorphism

$$\mathcal{A} \cong \mathcal{H}^* \otimes \mathcal{H} \tag{B.75}$$

in the category of Banach vector spaces.
As second interpretation we identify it with the Hermitian inner product on a GNS-constructed Hilbert space

$$\mathcal{H}_\omega := \mathcal{A}/\mathcal{J}_\omega \cong \mathcal{A} \quad \text{for } \omega = \frac{1}{n}\mathbb{1}, \tag{B.76}$$

which becomes isomorphic to the matrix Banach space itself, once the Gelfand ideal $\mathcal{J}_\omega$ turns out to be trivial in the case for maximal mixed state $\omega = \frac{1}{n}\mathbb{1}$. This is also the case in the more general situations for states with maximal rank (m=n),

$$\omega := \sum_{i=1}^{n} \lambda_i \left|i\right\rangle \left\langle i\right|. \tag{B.77}$$

Hence, we end up with a GNS-induced Hermitian tensor field, which is non-degenerate on the whole space $\mathcal{A}$, which in this way becomes a Kählerian manifold.

**Remark B.5.** *These observations may lead to consider the a family vector spaces*

$$\mathcal{A}_\alpha := (\mathcal{A}, \left\langle A\,|A\right\rangle_\alpha) \tag{B.78}$$

*with distinguished, in general degenerate inner products $\left\langle A\,|A\right\rangle_\alpha$ parametrized by the space of states $D(\mathcal{A})$ according to*

$$\bigcup_{\alpha\in D(\mathcal{A})} \mathcal{A}_\alpha. \tag{B.79}$$

*Via the GNS-construction over each state one finds*

$$\bigcup_{\alpha\in D(\mathcal{A})} \mathcal{A}_\alpha/\mathcal{J}_\alpha = \bigcup_{\alpha\in D(\mathcal{A})} \mathcal{H}_\alpha \cong_{loc} D(\mathcal{A}) \times \mathcal{H}_\alpha, \tag{B.80}$$

*and therefore a Kähler bundle over $D(\mathcal{A})$ (see also [61] and references therein). Note that a Kähler bundle is not a fiber bundle in the usual sense, since its 'fibers' $\mathcal{H}_\alpha$ are not isomorphic. The structure*

$$\langle dA \otimes dA \rangle_\alpha \tag{B.81}$$

*provides therefore a family of tensor fields on $\mathcal{A}$ being parametrized by $\alpha \in D(\mathcal{A})$.*

## B.4 Covariant tensors on $u^*(n)$ - Construction in the Bloch representation

Consider a decomposition of the vector space $\mathcal{A} \cong M_n(\mathbb{C})$ of complex matrices according to

$$\mathcal{A} = \mathrm{Re}\mathcal{A} + \mathrm{Im}\mathcal{A}, \tag{B.82}$$

with

$$M_n(\mathbb{C}) \cong \mathbb{R}^{n^2} \oplus i\mathbb{R}^{n^2} \cong u^*(n) \oplus iu^*(n), \tag{B.83}$$

into two subspaces of Hermitian and anti-Hermitian matrices by decomposing each element into real and imaginary part

$$A \equiv A_1 + iA_2, \quad A_k \in u^*(n), \tag{B.84}$$

with

$$A_1 := \frac{1}{2}(A + A^\dagger) \tag{B.85}$$

$$A_2 := \frac{1}{2i}(A - A^\dagger). \tag{B.86}$$

By introducing a basis $\{\sigma_j\}_{j\in J}$ of Hermitian matrices on the real subspaces $u^*(n)$ we identify coordinate functions

$$q^j(A_1) := \sigma_j(A_1) \equiv \mathrm{Tr}(\sigma_j A_1) \tag{B.87}$$

$$p^j(A_2) := \sigma_j(A_2) \equiv \mathrm{Tr}(\sigma_j A_2) \tag{B.88}$$

as particular instances of real-valued linear functionals on $u^*(n)$ defined by the introduced basis elements. Hence, we have for all $A_k \in u^*(n)$

$$q^j, p^j \in \mathrm{Lin}(u^*(n), \mathbb{R}), \tag{B.89}$$

where we shall note, that $q_1$ and $p_2$ are functionals on distinguished spaces appearing as two (direct summed) copies of $u^*(n)$.
By means of this functionals we recover the Bloch representation expansions

$$A_1 = q^0(A_1)\mathbb{1} + \sum_{j=1} q^j(A_1)\sigma_j \equiv \sum_{j=0} q^j(A_1)\sigma_j \tag{B.90}$$

$$A_2 = p^0(A_2)\mathbb{1} + \sum_{k=1} p^k(A_2)\sigma_k \equiv \sum_{j=0} p^j(A_2)\sigma_j, \tag{B.91}$$

where we set $\sigma_0 \equiv \mathbb{1}$. Within this coordinates, we identify the Hermitian matrix valued 1-forms

$$dA_1 = \sum_{j=0} dq^j(A_1)\sigma_j \tag{B.92}$$

$$dA_2 = \sum_{k=0} dp^k(A_2)\sigma_k \tag{B.93}$$

on the subspace $u^*(n)$ of Hermitian matrices, which gives rise to an operator valued 1-form

$$dA = dA_1 + idA_2 \tag{B.94}$$

on $\mathcal{A}$. In this way we may identify an operator valued covariant tensor field

$$dA^\dagger \otimes dA. \tag{B.95}$$

It decomposes into

$$dA_1 \odot dA_1 + dA_2 \odot dA_2 + idA_1 \wedge dA_2, \tag{B.96}$$

a symmetric and an anti-symmetric operator valued tensor (0,2)-tensor. Here we may consider the pull-back of the tensor field (B.96) from $\mathcal{A}$ to Re$\mathcal{A} = u^*(n)$, whenever we set $dA_2 \equiv 0$. In particular for $n = 2$ we find

$$\sigma_j dq^0 \odot dq^j + \sum_{j=0}^{3} \mathbb{1} dq^j \odot dq^j + i\sigma_l \epsilon_{jkl} dq^j \wedge dq^k. \tag{B.97}$$

An evaluation with an dual element $\omega \in \mathcal{A}^* \cong M_2(\mathbb{C})$ gives

$$\mathrm{Tr}(\omega\sigma_j) dq^0 \odot dq^j + \sum_{j=0}^{3} \mathrm{Tr}(\omega) dq^j \odot dq^j + i\mathrm{Tr}(\omega\sigma_l)\epsilon_{jkl} dq^j \wedge dq^k. \tag{B.98}$$

If we set

$$\omega \equiv A_1 \tag{B.99}$$

we find

$$q^j dq^0 \odot dq^j + 2q^0 dq^j \odot dq^j + iq^l \epsilon_{jkl} dq^j \wedge dq^k. \tag{B.100}$$

On the other hand, if $\omega$ is a multiple of the identity

$$\omega \equiv c\mathbb{1}, \tag{B.101}$$

we end up with an Euclidean tensor field. In particular for $c \equiv 1/n$ we have

$$\sum_{j=0}^{3} dq^j \odot dq^j. \tag{B.102}$$

## B.5 Tensors from $\mathcal{G}$-orbits - A relation to IOVTs

For a general submanifold $N$ being embedded in the vector space $\mathcal{A}$ of a $C^*$-algebra of finite dimensional operators, we will find for a given embedding

$$\iota : N \hookrightarrow \mathcal{A}, \tag{B.103}$$

a pull-back tensor field

$$\iota_N^* \langle dA \otimes dA \rangle_\omega \tag{B.104}$$

on $N$. In particular we will be interested in the following to consider cases where $N$ admits the structure of a homogenous space by means of orbits associated to a unitary representation of a Lie group $\mathcal{G}$ in $\mathcal{A}$. By considering for this purpose a family of actions of $\mathcal{G}$ on a given fiducial operator $A_0 \in \mathcal{A}$, we encounter several possibilities for realizing the embedding of these orbits in $\mathcal{A}$ . There will be indeed different representations of the group $\mathcal{G}$ generating the corresponding orbit $\mathcal{G}/\mathcal{G}_{A_0}$ in $\mathcal{A}$ by acting on this fiducial point transitively (all invariant actions to be identified are denoted here by the isotropy group $\mathcal{G}_{A_0}$). Hence, we have to distinguish different embedding of the orbit in dependence of the chosen representation. As a first fundamental difference we shall distinguish two classes provided by vector representation actions on the one hand, and adjoint representation actions on the other hand.

### B.5.1 Vector representation induced orbits

By taking into account the $n^2$-dimensional complex vector space structure of $\mathcal{A}$ we may identify vector representations

$$Q \to \mathrm{Aut}(\mathcal{A}) = GL(n^2, \mathbb{C}), \tag{B.105}$$

in particular by focusing on those given by unitary vector representations of a unitary subgroup $Q \subset U(n)$,

$$Q \to U(n^2) \subset GL(n^2, \mathbb{C}), \tag{B.106}$$

$$U(q)U(q') = U(q\, q'), \tag{B.107}$$

with $q, q' \in Q$.

**Remark B.6.** *The following consideration may become valid also for unitary* *representations of general Lie groups $\mathcal{G}$. For sake of simplicity however, we shall restrict in this and the following subsection to case of a unitary representation of a unitary subgroup $Q \subset U(n^2)$.*

The action on a fiducial point $A_0 \in \mathcal{A}$ reads

$$U(q)A_0 \equiv A_q \in \mathcal{A}, \quad q \in Q \tag{B.108}$$

and defines an embedding of the group manifold $Q$ in the dual algebra $\mathcal{A}$ by means of an orbit $Q/Q_{A_0}$. A particular simple embedding of $Q$ may be achieved, whenever we have the special setting given by $Q = U(n^2)$, $A_0 = \mathbb{1}$, and $U(q) = q$, the defining representation. The embedding action (B.108) reduces to

$$U(q)\mathbb{1} \equiv q \in \mathcal{A},\, q \in Q, \tag{B.109}$$

It is important to underline this special case, since we will encounter much more possibilities of realizing the embedding, once we have to deal with more generic situations given by $Q \subset U(n)$.
A homogeneous space $Q/Q_{A_0}$ which becomes embedded according to (B.108) in $\mathcal{A}$ induces a pulled back tensor

$$\iota_Q^* \left\langle dA \otimes dA \right\rangle_\omega = \mathrm{Tr}(\omega\, dA_q^\dagger \otimes dA_q) \tag{B.110}$$

on the Lie group $Q$.
To give an explicit expression of the differentials involved in the pulled back tensor, we need to specify a Lie algebra representation

$$R : T_eQ \to T_eU(n), \tag{B.111}$$

$$[R(e_j), R(e_k)] = R([e_j, e_k]), \tag{B.112}$$

associated to (B.106) where $\{e_j\}_{j\in J}$ denotes a basis of Hermitian matrices spanning the Lie algebra $T_eQ$. By the use of anti-Hermitian matrix valued left invariant 1-forms

$$U^\dagger(q)dU(q) \equiv iR(e_j)\theta^j(q) \tag{B.113}$$

composed out of a basis $\{\theta_j\}_{j\in J}$ of left invariant 1-forms on the group manifold and a basis $\{e_j\}_{j\in J}$ of Hermitian matrices spanning the tangent space $T_eU(n)$, one finds then

$$dU(q) = iU(q)R(e_j)\theta^j, \tag{B.114}$$

and therefore

$$dA_q = dU(q)A_0 = iU(q)R(e_j)\theta^j A_0. \tag{B.115}$$

In this way the pulled back tensor field reads

$$\mathrm{Tr}(\omega\, dA_q^\dagger \otimes dA_q) = \mathrm{Tr}(Ad_{A_0}(\omega)R(e_j)R(e_k))\theta^j \otimes \theta^k, \tag{B.116}$$

admitting a decomposition into a symmetric

$$\mathrm{Tr}(Ad_{A_0}(\omega)\,[R(e_j), R(e_k)]_+)\theta^j \odot \theta^k, \tag{B.117}$$

and an anti-symmetric tensor

$$i\mathrm{Tr}(Ad_{A_0}(\omega)\,[R(e_j), R(e_k)]_-)\theta^j \wedge \theta^k. \tag{B.118}$$

Here we shall identify the coefficients of the tensor in the above decomposition by means of

$$L_{jk} := \mathrm{Tr}(Ad_{A_0}(\omega)R(e_j)R(e_k)) \tag{B.119}$$

$$L_{(jk)} := \mathrm{Tr}(Ad_{A_0}(\omega)\,[R(e_j), R(e_k)]_+)) \tag{B.120}$$

$$L_{[jk]} := \mathrm{Tr}(Ad_{A_0}(\omega)\,[R(e_j), R(e_k)]_-). \tag{B.121}$$

**Remark B.7.** *The symmetric part of the tensor field may become related to a*

*Riemannian structure if $Ad_{A_0}(\omega)$ is a Hermitian operator which is positive. In general however, we may encounter also pseudo-Riemanian structures, whenever $Ad_{A_0}(\omega)$ turns out to be an element of the sub-algebra $u^*_{k_+,k_-}(n) \subset \mathcal{A}$ of Hermitian matrices with $k_+$ positive and $k_-$ negative eigenvalues. The relation of the of the sub-algebra $u^*_{k_+,k_-}(n)$ with semi-Hermitian structures has been described in [43].*

**Remark B.8.** *For $A_0 = \mathbb{1}$ we will have a trivial isotropy group $Q_{\mathbb{1}} = \{\mathbb{1}\}$. In this case the anti-symmetric tensor*

$$Tr(\omega\,[R(e_j), R(e_k)]_-)\theta^j \wedge \theta^k \tag{B.122}$$

*identifies a pull-back structure on the orbit $Q/Q_{\mathbb{1}} = Q$. It will vanish in dependence of the state $\omega$ in all directions $R(e_j)$ with*

$$[R(e_j), \omega]_- = 0 \tag{B.123}$$

*due to*

$$Tr(\omega\,[R(e_j), R(e_k)]_-) = Tr(R(e_k)\,[\omega, R(e_j)]_-). \tag{B.124}$$

*With other words the pulled back tensor on the orbit $Q/Q_{\mathbb{1}} = Q$ will be degenerate in these directions. On the corresponding quotient space $Q/Q_\omega$ however, with $Q_\omega$, an isotropy subgroup of $Q$ being generated by all Hermitian matrices $R(e_j)$ which commute with the state $\omega$, this tensor becomes related to a non-degenerate and therefore symplectic structure. Moreover, it coincides with the symplectic tensor*

$$Tr(\omega\, dU^\dagger \wedge dU), \tag{B.125}$$

*which has been related to the analysis on degeneracy structures of geometric phases for multi-level quantum systems discussed in [72].*

In particular we may distinguish between different types of pulled back tensors according to the three special cases

$$Ad_{A_0}(\omega) = \begin{cases} \omega & \text{for } A_0 = \mathbb{1} \\ A_0 A_0^\dagger & \text{for } \omega = \mathbb{1} \\ A_0^2 A_0^\dagger & \text{for } \omega = A_0. \end{cases} \tag{B.126}$$

Crucially, the first two cases provide a relation to tensor fields from quantum state-evaluated IOVTs on Lie groups $\mathcal{G}$ as discussed in section 4.5.1. Indeed, in the second case, $A_0 A_0^\dagger$ is a Hermitian operator, which is positive. Hence, by taking into account a normalizing factor $(\mathrm{Tr}(A_0 A_0^\dagger))^{-1}$ we shall be able to close a circle to the IOVT construction as done in section 4.5.1. This will be discussed for general Lie groups $\mathcal{G}$ in detail in subsection B.5.3.

### B.5.2 Adjoint representation induced orbits

The unitary representation $U(q) \in U(n)$ in (B.106) may induce a non-equivalent class of embeddings of orbits in $\mathcal{A}$ by means of the adjoint action

$$A_q \equiv U(q) A_0 U^\dagger(q), \tag{B.127}$$

on a fiducial point $A_0 \in A$. The corresponding isotropy group $Q_{A_0}$ becomes in this case defined by all $q \in Q$ for which

$$[U(q), A_0] = 0 \tag{B.128}$$

holds. The induced pulled back tensor on $Q \subset \mathcal{A}$ may read then again formally

$$\iota_Q^* \left\langle dA \otimes dA \right\rangle_\omega = \mathrm{Tr}(\omega \, dA_q^\dagger \otimes dA_q). \tag{B.129}$$

The evaluation of the coefficients however, will differ crucially from those in $(B.119) - (B.121)$ associated to the vector representation induced embedding

(B.108), as we will show in the following. With the differential $dA_q$ given by

$$dA_q = Ad_{U(q)}[U^\dagger(q)dU(q), A_0] \tag{B.130}$$

and the operator-valued left invariant 1-forms

$$U^\dagger(q)dU(q) = iR(e_j)\theta_j(q) \tag{B.131}$$

with $\{\theta_j\}_{j\in J}$, as defined in section 4.5.1 as a basis of left-invariant 1-forms on the Lie group manifold $Q$ (with a Lie-algebra admitting a basis $\{e_j\}_{j\in J}$) , one finds

$$\begin{aligned} &\mathrm{Tr}(\omega(U[iR(e_j), A_0]U^\dagger U[iR(e_k), A_0]U^\dagger))\theta^j \otimes \theta^k \\ &= -\mathrm{Tr}(Ad_{U(q)}(\omega)[R(e_j), A_0][R(e_k), A_0])\theta^j \otimes \theta^k. \end{aligned} \tag{B.132}$$

This tensor field shows clearly that it is degenerate along the commutant of $A_0$. Hence, it provides the pull-back tensor field from a non-degenerate structure on the homogeneous space $Q/Q_{A_0}$.
By focusing on the coefficients of the tensor, evaluated on the identity $q_0 \equiv \mathbb{1} \in Q$ with $U(q_0) = \mathbb{1}$ we find

$$K_{jk} := -\mathrm{Tr}(\omega[R(e_j), A_0][R(e_k), A_0]), \tag{B.133}$$

admitting via (anti)-symmetrization a decompostion

$$-\mathrm{Tr}(\omega[[R(e_j), A_0], [R(e_k), A_0]]_+) - i\mathrm{Tr}(\omega[[R(e_j), A_0], [R(e_k), A_0]]_-). \tag{B.134}$$

Hence, the tensor coefficients may denoted in the following form

$$K_{jk} = K_{(jk)} + iK_{[jk]}, \tag{B.135}$$

decomposed into a symmetric part

$$K_{(jk)} := -\mathrm{Tr}(\omega[[R(e_j), A_0], [R(e_k), A_0]]_+) \tag{B.136}$$

and an anti-symmetric part

$$K_{[jk]} := -\mathrm{Tr}(\omega[[R(e_j), A_0], [R(e_k), A_0]]_-). \tag{B.137}$$

**Remark B.9.** *The anti-symmetric part*

$$K_{[jk]} = -\,Tr([R(e_j), A_0]\,[\omega, [R(e_k), A_0]]_-) \tag{B.138}$$

*will vanish along all directions $[R(e_k), A_0]$ for which*

$$[\omega, [R(e_k), A_0]]_- = 0 \tag{B.139}$$

*holds. It defines therefore a symplectic structure on the quotient space $(Q/Q_{A_0})/Q_\omega$ with $Q_\omega$, the isotropy group generated by the corresponding invariant directions $[R(e_k), A_0]$. For the special case $\omega = \mathbb{1}$ one finds therefore only a symmetric part*

$$K_{jk} = K_{(jk)} = -\,Tr([R(e_j), A_0][R(e_k), A_0]), \tag{B.140}$$

*which defines a left-invariant metric [72].*

The tensor coefficients described in (B.133)-(B.137) may extract local properties of the orbits. Indeed, by considering variational aspects of these tensors in the neighborhood of $q_0 = \mathbb{1}$, we may find also global properties of the orbits encoded by higher rank tensors, like for instance the Riemanian curvature associated to a Riemannian metric (B.140). Anti-symmetric tensors which are symplectic on the other hand, may be used for the constructions of volume-forms. More general, by applying $k$ times the tensor product on the differential (B.130)

$$dA_q = Ad_{U(q)}[iR(e_j), A_0]\theta_j(q) \tag{B.141}$$

of the matrix valued function $A_q$ on the orbit, we may consider in this regard a (0,k)-tensor construction

$$\bigotimes_{l=l_1}^{l_k} dA_q = (i)^k U(q) \prod_{l_1}^{l_k} [R(e_l), A_0] U(q)^\dagger \bigotimes_{l=l_1}^{l_k} \theta^l \tag{B.142}$$

and its evaluation on the tangent spaces according to

$$(i)^k \operatorname{Tr}\left( Ad_{U(q)}(\omega) \prod_{l=l_1}^{l_k} [R(e_l), A_0] \right) \bigotimes_{l=l_1}^{l_k} \theta^l. \tag{B.143}$$

In this way we may identify for instance the coefficients of a scalar valued (0,4)-tensor field defined globally on the orbit by

$$\operatorname{Tr}(Ad_{U(q)}(\omega)[R(e_l), A_0][R(e_r), A_0][R(e_j), A_0][R(e_k), A_0]). \tag{B.144}$$

The later reduces to

$$K_{lrjk} := \operatorname{Tr}(\omega[R(e_l), A_0][R(e_r), A_0][R(e_j), A_0][R(e_k), A_0]), \tag{B.145}$$

when evaluated over $q_0 = \mathbb{1}$.

### B.5.3 A relation to IOVTs on $\mathcal{G}$: The case $\omega = \mathbb{1}$

Let us restart by considering an inner product

$$c\operatorname{Tr}(\omega A^\dagger A) \tag{B.146}$$

on a $C^*$-matrix algebra $\mathcal{A} \cong M_n(\mathbb{C})$ which becomes identified with a Hilbert manifold of Hilbert Schmidt operators in finite dimensions for $\omega = \mathbb{1}$, where $c \in \mathbb{R}_0$ denotes a normalization constant. With (B.55), we may promote this inner product to a tensor field

$$c\mathrm{Tr}(\omega dA^\dagger \otimes dA), \tag{B.147}$$

on the vector space $\mathcal{A}$. For $\omega = \mathbb{1}$ we find:

**Theorem B.10.** *Let $\{\theta_j\}_{j\in J}$ a basis of left-invariant 1-forms on $\mathcal{G}$, and $\{X_j\}_{j\in J}$, a Lie algebra basis of $\mathcal{G}$ with $\{iR(X_j)\}_{j\in J}$, its representation in the Lie-Algebra of $U(\mathcal{H})$ associated to a unitary representation $U : \mathcal{G} \to U(\mathcal{H})$. The submersion*

$$\iota : \mathcal{G} \to \mathcal{A} \tag{B.148}$$

*defined by the action*

$$A_g \equiv \begin{cases} U(g)A_0U(g)^{-1}, & \textit{(adjoint represention)} \\ \\ U(g)A_0, & \textit{(vector representation)} \end{cases} \tag{B.149}$$

*induces then a pull-back of the tensor field*

$$\beta := c\,Tr(dA^\dagger \otimes dA) \tag{B.150}$$

*from $\mathcal{A}$ to $\mathcal{G}$, according to*

$$\iota_{\mathcal{G}}^*\beta = \begin{cases} c\,Tr([R(X_j), A_0^\dagger][R(X_k), A_0])\theta^j \otimes \theta^k, \\ \\ c\,Tr(A_0A_0^\dagger R(X_k)R(X_k))\theta^j \otimes \theta^k \end{cases} \tag{B.151}$$

*associated to the adjoint and the vector representation acting on $A_0 \in \mathcal{A} \cong$*

$M_n(\mathbb{C})$ *respectively.*

*Proof.* The pull-back of the tensor field

$$\iota_{\mathcal{G}}^*\beta = c\mathrm{Tr}(d\iota_{\mathcal{G}}^*(A^\dagger) \otimes d\iota_{\mathcal{G}}^*(A)) \tag{B.152}$$

is provided by the pull-back of a function $A \in \mathrm{Lin}(\mathcal{A})$ on the dual algebra $\mathcal{A}$ to a function

$$\iota_{\mathcal{G}}^*(A) \equiv A_{(\bullet)} : g \mapsto A_g \tag{B.153}$$

on the Lie group manifold $\mathcal{G}$. The pull-back of the function is induced by the immersion of the Lie-group in dependence of the chosen action in (B.149). Hence, the pull-back tensor is given by evaluating

$$c\mathrm{Tr}(dA_g^\dagger \otimes dA_g). \tag{B.154}$$

Let us begin with the adjoint representation, where we consider the relation $d(U(g)U(g)^{-1}) = 0$, resp.

$$dU(g)^{-1} = -U(g)^{-1}dU(g)U(g)^{-1}, \tag{B.155}$$

within the exterior derivative

$$\begin{aligned} d\rho_g &= dU(g)A_0U(g)^{-1} + U(g)A_0dU(g)^{-1} \\ &= dU(g)A_0U(g)^{-1} - U(g)A_0U(g)^{-1}dU(g)U(g)^{-1}. \end{aligned} \tag{B.156}$$

By multiplication with the identity on the first summand we get

$$\begin{aligned} U(g)U(g)^{-1}dU(g)A_0U(g)^{-1} - U(g)A_0U(g)^{-1}dU(g)U(g)^{-1} \\ = U(g)([iR(X_j)\theta^j, A_0])U(g)^{-1}, \end{aligned}$$

where we identify the left-invariant operator valued-1-forms $U(g)^{-1}dU(g) \equiv$

$iR(X_j)\theta^j$. Furthermore we find in the last expression the adjoint representation

$$\mathrm{Ad}_{U(g)}[iR(X_j)\theta^j, A_0]. \tag{B.157}$$

Since $\theta^j$ is scalar-valued, we may reorganize the operator valued 1-form $d\rho_g$ into

$$dA_g = \mathrm{Ad}_{U(g)}[iR(X_j), A_0]\theta^j. \tag{B.158}$$

The pull-back tensor $c\mathrm{Tr}(dA_g^\dagger \otimes dA_g)$ reads then

$$c\mathrm{Tr}(\mathrm{Ad}_{U(g)}[-iR(X_j), A_0^\dagger]\theta^j \otimes \mathrm{Ad}_{U(g)}[iR(X_k), A_0]\theta^k). \tag{B.159}$$

By defining $\otimes$ on the $\theta^k$-spanned module of 1-forms $\mathcal{T}_1^0(\mathcal{G})$ being related via the pull-back to the tensor product on $F := \mathcal{T}_1^0(u^*(\mathcal{H}))$, we find

$$\begin{aligned} &c\mathrm{Tr}(\mathrm{Ad}_{U(g)}[R(X_j), A_0^\dagger]\mathrm{Ad}_{U(g)}[R(X_k), A_0])\theta^j \otimes \theta^k \\ &= c\mathrm{Tr}(\mathrm{Ad}_{U(g)}([R(X_j), A_0^\dagger][R(X_k), A_0]))\theta^j \otimes \theta^k. \end{aligned} \tag{B.160}$$

By using

$$\mathrm{Tr}(\mathrm{Ad}_U C) = \mathrm{Tr}(U(CU^{-1})) = \mathrm{Tr}((CU^{-1})U) = \mathrm{Tr}(C),$$

for any complex matrix $C$, we end up with the statement for the case of an adjoint representation induced immersion.
In the case of a vector representation induced immersion we have to consider the exterior derivative

$$dA_g = dU(g)A_0 = U(g)U^{-1}(g)dU(g)A_0 = iU(g)R(X_j)A_0\theta^j \tag{B.161}$$

and its tensor product yielding

$$dA_g^\dagger \otimes dA_g = A_0^\dagger R(X_j)R(X_k)A_0\theta^j \otimes \theta^k \tag{B.162}$$

and therefore the evaluation

$$c\mathrm{Tr}(A_0^\dagger R(X_j)R(X_k)A_0)\theta^j \otimes \theta^k. \tag{B.163}$$

$\square$

**Corollary B.11.** *Let $T^\alpha_{jk}$ and $T^\beta_{jk}$ denote the coefficients of the pull-back tensor on a Lie group $\mathcal{G}$ obtained by a corresponding unitary action on $(\mathcal{H}, \alpha)$ according to [3, 55] and into $(\mathcal{A}, \beta)$ according to Theorem B.10 respectively. Up to a normalization constant $c \in \mathbb{R}$, one finds:*

*(a) If $A_0$ is Hermitian in the adjoint representation then*

$$T^\beta_{jk} = T^\beta_{kj}. \tag{B.164}$$

*(b) If $A_0$ is a normalized rank-1 projector in the adjoint representation then*

$$T^\beta_{(jk)} = T^\alpha_{(jk)} \tag{B.165}$$

*(c) If $A_0$ is a normalized rank-1 projector in the vector representation then*

$$T^\beta_{[jk]} = T^\alpha_{[jk]} \tag{B.166}$$

*Proof.* (a) It is clear that the coefficients

$$c\mathrm{Tr}([R(X_j), A_0^\dagger][R(X_k), A_0]) \tag{B.167}$$

are symmetric in the indices $j$ and $k$ due to the permutation invariance within the trace if $A_0^\dagger = A_0$.
(b) The expansion shows that these coefficients may be seen furthermore

decomposed into a sum of two terms[17]

$$\mathrm{Tr}(A_0^2[R(X_j), R(X_k)]_+) - \mathrm{Tr}(A_0 R(X_j) A_0 R(X_k)), \tag{B.168}$$

where we set the normalization constant $c$ equal $-1/2$. At this point we chose the fiducial matrix to be pure, i.e. $A_0^2 = A_0$, say

$$A_0 := |0\rangle \langle 0| \in u^*(\mathcal{H}), \tag{B.169}$$

and find

$$\mathrm{Tr}(A_0[R(X_j), R(X_k)]_+) - \mathrm{Tr}(A_0 R(X_j))\mathrm{Tr}(A_0 R(X_k)). \tag{B.170}$$

(c) For the anti-symmetrized pull-back coefficients in the vector representation we have

$$c\mathrm{Tr}(A_0^2[R(X_j), R(X_k)]_-) \tag{B.171}$$

Hence, the second statement is immediate. □

In conclusion, we have found a generalization of the symmetric and (anti-symmetric) part of the pull-back tensor obtained from the Hilbert space. This tensor field may become non-degenarate and therefore Riemannian (symplectic), whenever it descends from the Lie group $\mathcal{G}$ to a unitarily generated orbit $\mathcal{G}/\mathcal{G}_0$ in $\mathcal{A}$ associated to a fiducial Hermtitian matrix and an unitary adjoint (vector) representation action $\phi : \mathcal{G} \times \mathcal{A} \to \mathcal{A}$ with isotropy group $\mathcal{G}_0 := \{g \in \mathcal{G} | \phi(g, A_0) = \rho_g = A_0 \in \mathcal{A}\}$. For a possible application finally, we will need:

**Corollary B.12** (Relation to IOVTs). *The fiducial operator $A_0$-dependent pull-back of*

$$\beta = c\,Tr(dA^\dagger \otimes dA), \tag{B.172}$$

[17] We distinguish the conventions $[A, B]_+ = \frac{1}{2}(AB + BA)$, $[A, B]_- = \frac{1}{2i}(AB - BA)$, and $[A, B] = AB - BA$.

*to a unitarily represented Lie group $\mathcal{G}$, coincides in the vector representation with an operator valued tensor field (IOVT) evaluated on a linear functional*

$$\rho := cA_0A_0^\dagger \in \mathcal{A}^* \tag{B.173}$$

*which is*

$$\begin{cases} \textit{positive for } c = 1 \\ \\ \textit{positive and normalized for } c = \frac{1}{Tr(A_0^\dagger A_0)} \end{cases} \tag{B.174}$$

*Proof.* The vector representation action induces on $\mathrm{Tr}(A_0^\dagger A_0)$ the transformation

$$\mathrm{Tr}((U(g)A_0)^\dagger U(g)A_0) = \mathrm{Tr}(A_0^\dagger U(g)^\dagger U(g)A_0) = \mathrm{Tr}(A_0^\dagger A_0). \tag{B.175}$$

By comparing the operator valued tensor in (4.76) with

$$\iota_{\mathcal{G}}^*\beta = c\mathrm{Tr}(A_0A_0^\dagger R(X_k)R(X_k))\theta^j \otimes \theta^k, \tag{B.176}$$

we may set $\rho := cA_0A_0^\dagger$ for any $A_0 \in \mathcal{A} \cong M_n(\mathbb{C})$. $\square$

# C Tensors from generalized momentum maps

While the Fubini-Study metric identifies a quantum information metric for pure states [7], we may try to approach possible quantum information metrics also for the generalized regime of mixed states *as seen form the Hilbert space.* Let us for this purpose generalize constructively the notion of a pure state to a mixed state $\rho$ by means of a convex combination of pure states $\rho_{\psi_l} := \rho_l$, according to

$$\rho := \sum_l p_l\rho_l, \text{ and } \sum_l p_l = 1. \tag{C.1}$$

This suggests to consider a generalized momentum map

$$\tilde{\mu} : \bigoplus_j \mathcal{H}_0 \longrightarrow u^*(\mathcal{H}), \tag{C.2}$$

$$\tilde{\mu}(|\psi_1\rangle \oplus |\psi_2\rangle \oplus ...) = \sum_l p_l \mu(|\psi_l\rangle) = \sum_l p_l \frac{|\psi_l\rangle \langle\psi_l|}{\langle\psi_l |\psi_l\rangle} := \rho, \tag{C.3}$$

which is defined on a direct sum of Hilbert spaces. After considering the pull-back tensor

$$\frac{1}{2}\mathrm{Tr}(d\rho \otimes d\rho) = \frac{1}{2}\sum_{l,s} p_l p_s \mathrm{Tr}(d\frac{|\psi_l\rangle \langle\psi_l|}{\langle\psi_l |\psi_l\rangle} \otimes d\frac{|\psi_l\rangle \langle\psi_l|}{\langle\psi_l |\psi_l\rangle}) \tag{C.4}$$

by means of the differentials

$$d\frac{|\psi_l\rangle \langle\psi_l|}{\langle\psi_l |\psi_l\rangle} = \frac{|d\psi_l\rangle \langle\psi_l| + |\psi_l\rangle \langle d\psi_l|}{\langle\psi_l |\psi_l\rangle} - \frac{|\psi_l\rangle \langle\psi_l| \, d\langle\psi_l |\psi_l\rangle}{\langle\psi_l |\psi_l\rangle^2}, \tag{C.5}$$

in the expanded expression

$$\begin{aligned}
\sum_{l,s} p_l p_s \Bigg( & \frac{\langle d\psi_s \otimes d\psi_l\rangle}{\langle\psi_l |\psi_l\rangle \langle\psi_s |\psi_s\rangle} \langle\psi_l |\psi_s\rangle + \frac{\langle d\psi_l \otimes d\psi_s\rangle}{\langle\psi_l |\psi_l\rangle \langle\psi_s |\psi_s\rangle} \langle\psi_s |\psi_l\rangle \\
& - \frac{\langle\psi_s |d\psi_l\rangle \otimes d\langle\psi_s |\psi_s\rangle}{2\langle\psi_l |\psi_l\rangle \langle\psi_s |\psi_s\rangle^2} \langle\psi_l |\psi_s\rangle - \frac{\langle d\psi_l |\psi_s\rangle \otimes d\langle\psi_s |\psi_s\rangle}{2\langle\psi_l |\psi_l\rangle \langle\psi_s |\psi_s\rangle^2} \langle\psi_s |\psi_l\rangle \\
& - \frac{d\langle\psi_l |\psi_l\rangle \otimes \langle\psi_l |d\psi_s\rangle}{2\langle\psi_s |\psi_s\rangle \langle\psi_l |\psi_l\rangle^2} \langle\psi_s |\psi_l\rangle - \frac{d\langle\psi_l |\psi_l\rangle \otimes \langle d\psi_s |\psi_l\rangle}{2\langle\psi_s |\psi_s\rangle \langle\psi_l |\psi_l\rangle^2} \langle\psi_l |\psi_s\rangle \\
& + \frac{d\langle\psi_l |\psi_l\rangle \otimes d\langle\psi_s |d\psi_s\rangle}{2\langle\psi_s |\psi_s\rangle^2 \langle\psi_l |\psi_l\rangle^2} \langle\psi_l |\psi_s\rangle \langle\psi_s |\psi_l\rangle \\
& + \frac{\langle\psi_s |d\psi_l\rangle \otimes \langle\psi_l |d\psi_s\rangle}{2\langle\psi_l |\psi_l\rangle \langle\psi_s |\psi_s\rangle} + \frac{\langle d\psi_l |\psi_s\rangle \otimes \langle d\psi_s |\psi_l\rangle}{2\langle\psi_l |\psi_l\rangle \langle\psi_s |\psi_s\rangle} \Bigg),
\end{aligned} \tag{C.6}$$

one may impose the constraint

$$\langle\psi_l\,|\psi_s\rangle - \delta_{ls} = 0, \tag{C.7}$$

yielding $\langle d\psi_l\,|\psi_s\rangle = -\langle\psi_l\,|d\psi_s\rangle$, and find

$$\sum_l p_l^2\left(\frac{\langle d\psi_l \otimes d\psi_l\rangle}{\langle\psi_l\,|\psi_l\rangle} - \frac{\langle\psi_l\,|d\psi_l\rangle \otimes \langle d\psi_l\,|\psi_l\rangle}{2\,\langle\psi_l\,|\psi_l\rangle^2} - \frac{\langle d\psi_l\,|\psi_l\rangle \otimes \langle\psi_l\,|d\psi_l\rangle}{2\,\langle\psi_l\,|\psi_l\rangle^2}\right)$$
$$+\sum_{l\neq s} p_l p_s\left(\frac{\langle\psi_s\,|d\psi_l\rangle \otimes \langle\psi_l\,|d\psi_s\rangle}{\langle\psi_l\,|\psi_l\rangle\,\langle\psi_s\,|\psi_s\rangle}\right). \tag{C.8}$$

For pure states, this tensor field ‘collapses’ to the Fubini-Study metric as seen from the Hilbert space in (2.11).

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